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Abstract

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ON THE ABSOLUTE CONVERGENCE OF A POWER SERIES IN TWO VARIABLES

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Let a power series in two variables x, y be given:

$$\sum_{i=0,\ k=0}^{\infty} a_{ik} x^i y^k. \tag{1}$$

We shall say that the series (1) converges at the point (x_0, y_0) if

$$\lim_{\substack{p\to\infty \\ q\to\infty}} \sum_{i=0,\ k=0}^{p,q} a_{ik} x_0^i y_0^k = A < \infty.$$

In the case of a power series in one variable $\sum_{i=0}^{\infty} a_i x^i$, Abel' s theorem is valid, stating that from the convergence of the series at the point $x = x_0 \neq 0$ there follows its absolute convergence in the interval $|x| < |x_0|$. A similar result does not hold in the case of a power series in two variables.

Indeed, let

$$P(x) = \prod_{s=1}^{m} (x - x_s) = b_0 + b_1 x + \cdots + b_m x^m, \qquad x_l \neq x_k, \quad \text{if } l \neq k.$$

Put in the series (1)

$$a_{ik} = 0, \quad \text{if } i > m; \qquad a_{ik} = k! b_i, \quad \text{if } i \leq m.$$

If $p > m$, then

$$S_{p,q}(x, y) = \sum_{i,k=0}^{p,q} a_{ik} x^i y^k = \sum_{i,k=0}^{m,q} k! b_i x^i y^k = P(x) \sum_{k=0}^{q} k! y^k.$$

It follows from this equality that the series converges to zero on the straight lines $x = x_s$, $s = 1, 2, \dots, m$, since $P(x_s) = 0$, converges on the straight line $y = 0$, and diverges at the other points of the plane.

The example just considered shows, in particular, that a power series in two variables may converge at points of the plane not lying on the straight lines $x = 0$, $y = 0$, and may have no interior points of convergence. Hence follows the erroneousness of Osgood' s assertion in (1) (p. 33, lines 3-6) on the absolute convergence of the series (1) in the rectangle $\{|x| < |x_0|,\ |y| < |y_0|\}$, if it converges at the point (x_0, y_0).

In this note, conditions are indicated that ensure the validity of such an assertion.

Lemma. If the power series $\sum_{i,k=0}^{\infty} a_{ik} x^i y^k$ with coefficients $a_{i,k} = 0$, when $i > i_0$ and $k > k_0$, converges at the points $(x_n, y_n)_0^{n_0}$, $x_\nu \neq x_\omega$, $y_\nu \neq y_\omega$ for $\nu \neq \omega$, $n_0 = \max\{i_0, k_0\}$, then the inequalities

$$|a_{ik} x_{n_1}^i y_{n_2}^k| \leq M,$$

where

$$|x_{n_1}| = \min_{0 \leq n \leq n_0} |x_n|, \qquad |y_{n_2}| = \min_{0 \leq n \leq n_0} |y_n| \tag{2}$$

hold.

Proof. Since the given series converges at the points $(x_n, y_n)_0^{n_0}$, the sequences of partial sums of this series $S_{p,q}(x_n, y_n)$ and $S_{p-1,q}(x_n, y_n)$ have equal finite limits as $p, q \to \infty$. Consequently,

$$\lim_{\substack{p\to\infty \\ q\to\infty}} \{S_{p,q}(x_n, y_n) - S_{p-1,q}(x_n, y_n)\} = 0,$$

i.e.

$$x_n^p \sum_{k=0}^{k_0} a_{pk} y_n^k = x_n^p P_{k_0,p}(y_n) \to 0. \tag{3}$$

We write the polynomial $\sum_{k=0}^{k_0} a_{pk} y^k$ in the form of an interpolation polynomial with nodes at the points $\{y_n\}_0^{k_0}$,

$$\sum_{k=0}^{k_0} a_{pk} y^k = \sum_{n=0}^{k_0} P_{k_0,p}(y_n) \frac{\omega(y)}{(y - y_n)\omega'(y_n)}.$$

Since

$$\frac{\omega(y)}{(y-y_n)\omega'(y_n)} \equiv \sum_{k=0}^{k_0} c_{kn} y^k,$$

we have

$$\sum_{k=0}^{k_0} a_{pk} y^k = \sum_{n=0}^{k_0} P_{k_0,p}(y_n) \sum_{k=0}^{k_0} c_{kn} y^k.$$

Hence

$$a_{pk} = \sum_{n=0}^{k_0} P_{k_0,p}(y_n) c_{kn}, \qquad 0 \le k \le k_0. \tag{4}$$

From relations (2) and (3) it follows that

$$x_{n_1}^p P_{k_0,p}(y_n) \xrightarrow[p\to\infty]{} 0. \tag{5}$$

Multiplying both sides of equality (4) by $x_{n_1}^p$ and taking (5) into account, we shall have:

$$|a_{pk} x_{n_1}^p| \le \sum_{n=0}^{k_0} |P_{k_0,p}(y_n) x_{n_1}^p| \, |c_{kn}| \le$$

$$\le \left\{ \max_{\substack{0\le n\le k_0 \\ 0\le k\le k_0}} |c_{kn}| \right\} \sum_{n=0}^{k_0} |P_{k_0,p}(y_n) x_{n_1}^p| \xrightarrow[p\to\infty]{} 0.$$

If k is fixed, then $y_{n_2}^k$ is a constant quantity; therefore

$$a_{pk} x_{n_1}^p y_{n_2}^k \xrightarrow[p\to\infty]{} 0, \qquad 0 \le k \le k_0.$$

It is proved similarly that

$$a_{iq} x_{n_1}^i y_{n_2}^q \longrightarrow 0, \qquad 0 \le i \le i_0.$$

Since $a_{ik} = 0$ for $i > i_0$ and $k > k_0$, the lemma is proved.

Theorem. *If the power series (1) converges at the points* $(x_n, y_n)_1^\infty$*,* $|x_n| \downarrow x_0 > 0$*,* $|y_n| \downarrow y_0 > 0$*, then the given series converges absolutely in the rectangle* $|x| \leq x_0$*,* $|y| \leq y_0$*.*

Proof. Since the series (1) converges at the point (x_1, y_1), its general term $a_{ik}x_1^i y_1^k$ tends to zero as $i, k \to \infty$. Consequently, there exist indices i_0 and k_0 such that the inequalities

$$|a_{ik}x_1^i y_1^k| \leq 1, \qquad i > i_0, \qquad k > k_0. \tag{6}$$

will hold.

Since all terms of the series

$$\sum_{i=i_0+1,\ k=k_0+1}^{\infty} a_{ik}x^i y^k \tag{7}$$

are bounded in modulus by the number 1 at the point (x_1, y_1), it follows, as in the case of a power series in one variable, that the series (7) converges absolutely in the rectangle $|x| < |x_1|$, $|y| < |y_1|$. In particular, this series will converge absolutely on the set $\{(x_n, y_n)_2^\infty\}$. Since the original series also converges on this same set, the series

$$\sum_{i=0,\ k=0}^{\infty} a_{ik}x^i y^k - \sum_{i=i_0+1,\ k=k_0+1}^{\infty} a_{ik}x^i y^k, \tag{8}$$

whose construction is the same as that of the series considered in the lemma, will converge on it. By virtue of the lemma and the conditions of the theorem, from the convergence of the series (8) on the set $\{(x_n, y_n)_2^{n_0+2}\}$, $n_0 = \max\{i_0, k_0\}$, it follows that

$$|a_{ik}x_{n_0+2}^i y_{n_0+2}^k| \leq M. \tag{9}$$

Since $|x_{n_0+2}| < |x_1|$, $|y_{n_0+2}| < |y_1|$, the inequalities (6) and (9) imply the inequalities

$$|a_{ik}x_{n_0+2}^i y_{n_0+2}^k| \leq M + 1, \qquad i, k \geq 0.$$

Consequently, the series (1) converges absolutely in the rectangle $|x| < |x_{n_0+2}|$, $|y| < |y_{n_0+2}|$. This completes the proof of the theorem.

Corollary 1. *If the series (1) converges in some neighborhood of the point* (x_0, y_0)*,* $x_0 \neq 0$*,* $y_0 \neq 0$*, then it converges absolutely in the rectangle* $|x| \leq |x_0|$*,* $|y| \leq |y_0|$*.*

Corollary 2. *If the series (1) converges in an open domain G, then it converges absolutely in it.*

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REFERENCES

1. W. Osgood, *Lehrbuch der Funktionentheorie*, **2**, 1924.

Note: Figure translations are in progress. See original paper for figures.