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# MATHEMATICS

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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON AN ESTIMATE OF THE ACCURACY OF APPROXIMATE EIGENVALUES AND EIGENVECTORS**

*(Presented by Academician G. I. Petrov on 21 XI 1962)*

1. Consider the equation

$$T(\lambda)\varphi \equiv [E - K(\lambda)]\varphi = 0, \tag{1,1}$$

where  $K(\lambda)$  is a completely continuous linear operator in the Hilbert space  $H$ , depending analytically in norm on the parameter  $\lambda$  in some domain  $D$  of the complex  $\lambda$ -plane, and  $E$  is the identity operator <sup>(1)</sup>.

Let  $u \in H$  be a normalized approximate eigenvector of equation (1,1), obtained by some approximate method. Apply to equation (1,1) the generalized Galerkin method <sup>(2,3)</sup> in the first approximation, taking as the first basis vectors the vector  $u$  and some normalized vector  $v \in H$ . For the approximate eigenvalues we obtain the equation

$$E(\lambda) \equiv (T(\lambda)u, v) = 0. \tag{1,2}$$

Now let  $H_f$  denote the orthogonal complement of the vector  $f \in H$  in  $H$ , and let the operator  $P$  orthogonally project  $H$  onto  $H_v$ . Every vector  $\varphi \in H$  is represented in the form

$$\varphi = au + \omega, \tag{1,3}$$

where  $a = (\varphi, u)$ ,  $(\omega, u) = 0$ . Projecting  $T(\lambda)\varphi$  successively onto  $v$  and  $H_v$ , we replace (1,1) by the system of equations

$$(T(\lambda)\varphi, v) \equiv a(T(\lambda)u, v) + (T(\lambda)\omega, v) = 0; \tag{1,4}$$

$$PT(\lambda)\varphi \equiv aPT(\lambda)u + PT(\lambda)\omega = 0. \tag{1,5}$$

Rewrite (1,5) in the form

$$P[E - K(\lambda)]\omega = -aPT(\lambda)u. \quad (1,6)$$

Introducing the notation  $P\omega = \Omega$  and taking into account that  $\omega \in H_u$ , we obtain

$$\omega = A\Omega \equiv \Omega - \frac{(\Omega, u)}{(v, u)}v. \quad (1,7)$$

Passing now in (1,6) from  $\omega$  to  $\Omega$ , we shall have

$$[E - \tilde{K}(\lambda)]\Omega = -aPT(\lambda)u, \quad (1,8)$$

where the operator  $\tilde{K}(\lambda)$  assigns to the vector  $f \in H$  the vector  $\tilde{K}(\lambda)f \in H_v$  by the formula

$$\tilde{K}(\lambda)f = PK(\lambda)f - \frac{(f, u)}{(v, u)}PK(\lambda)v \quad (1,9)$$

and  $\tilde{K}(\lambda)$  may be considered in  $H_v$ , as we shall do everywhere below.

Let the approximate eigenvalue  $\lambda_1$  be a regular point for the operator  $\tilde{K}(\lambda)$ , i.e., there exists the operator  $[E - \tilde{K}(\lambda_1)]^{-1}$ , and consequently, in some neighborhood of the point  $\lambda_1$  there exists the operator  $[E -$

$-\tilde{K}(\lambda)]^{-1}$ . If an eigenvalue  $\lambda$  of equation (1,1) falls into this neighborhood, then, as is not hard to show, it is simple,  $a \neq 0$ , and (1,8) is solved in the following way:

$$\Omega = -a[E - \tilde{K}(\lambda)]^{-1}PT(\lambda)u. \quad (1,10)$$

If now we substitute (1,10) into (1,7), and (1,7) into (1,4), and divide by  $a$ , we obtain an equation satisfied by the eigenvalues of equation (1,1) that are regular points for  $\tilde{K}(\lambda)$ , and only by them, in the form

$$F(\lambda) + \Phi(\lambda) = 0, \quad (1,11)$$

where

$$\Phi(\lambda) = \frac{1}{a}(T(\lambda)\omega, v), \quad (1,12)$$

and  $\omega$  is determined from (1,7) and (1,10). Henceforth, by eigenvalues we shall mean only the roots of equation (1,11).

In order to estimate the closeness of the approximate and exact eigenvalues, one must in fact compare the zeros of the analytic functions  $F(\lambda)$  and  $F(\lambda) + \Phi(\lambda)$ .

The preceding arguments generalize the analogous arguments of G. I. Petrov <sup>(4)</sup>. The method of comparing the zeros of  $F(\lambda)$  and  $F(\lambda) + \Phi(\lambda)$  proposed by him reduces to the fact that in the equality  $|F(\lambda)| = |\Phi(\lambda)|$ , which is evident from (1,11), the right-hand side is majorized by a certain function of  $|\lambda|$ , after which  $|\lambda|$  is replaced by  $|\lambda_1| + |\lambda - \lambda_1|$ , and the resulting inequality is solved with respect to  $|\lambda - \lambda_1|$ . In order that the solution give an upper bound for  $|\lambda - \lambda_1|$ , it is necessary to impose severe additional conditions (for details, see <sup>(4)</sup>).

We shall use Rouché's theorem <sup>(5)</sup> to compare the zeros. This will make it possible to relax the conditions needed for carrying out the estimate.

2. For simplicity we shall consider a linear dependence of  $K(\lambda)$  on  $\lambda$ ,

$$K(\lambda) = M + \lambda N, \quad (2,1)$$

where  $M$  and  $N$  are completely continuous linear operators in  $H$ , and  $D$  coincides with the whole  $\lambda$ -plane. In the general case the arguments are carried out analogously. The first approximation by the Galerkin–Petrov method in the case (2,1) gives the single approximate eigenvalue

$$\lambda_1 = \frac{(u - Mu, v)}{(Nu, v)}, \quad (2,2)$$

where

$$F(\lambda) = (\lambda - \lambda_1)(Nu, v). \quad (2,3)$$

In accordance with Rouché's theorem, if in the  $\lambda$ -plane there exists a closed contour  $C$  enclosing the point  $\lambda_1$ , on which

$$|F(\lambda)| > |\Phi(\lambda)|, \quad (2,4)$$

then inside  $C$  there is a unique eigenvalue.

To apply Rouché's theorem, it is necessary to estimate the right-hand side in (2,4). The inequality, evident from (1,12) and (1,7),

$$|\Phi(\lambda)| \leq \|T(\lambda)A\| \cdot \|\Omega\| \quad (2,5)$$

(here and below, the operators standing under the norm sign are considered only in  $H_\nu$ ) reduces the matter to estimating  $\|\Omega\|$ .

It is easy to see that

$$\tilde{K}(\lambda) = \tilde{K}(\lambda_1) + (\lambda - \lambda_1)\tilde{N}. \quad (2,6)$$

where  $\tilde{N}$  is obtained from  $N$  in the same way as  $\tilde{K}(\lambda)$  from  $K(\lambda)$ . Represent  $\tilde{K}(\lambda_1)$  in the form

$$\tilde{K}(\lambda_1) = K_1 + K_2, \quad (2,7)$$

so that  $(E - K_1)^{-1}$  exists and is known to us. As  $K_1$  one may take, for example, a degenerate operator of sufficiently high order. In this case  $\|K_2\|$  can be made arbitrarily small.

Applying the operator  $(E - K_1)^{-1}$  to equation (1,8), taking into account (2,6) and (2,7), and introducing the notation

$$R = (E - K_1)^{-1}K_2, \quad S = (E - K_1)^{-1}\tilde{N}, \quad (2,8)$$

we obtain

$$[E - R - (\lambda - \lambda_1)S]\Omega = -a(E - K_1)^{-1}PT(\lambda)u. \quad (2,9)$$

Under the condition

$$\|R + (\lambda - \lambda_1)S\| < 1 \quad (2,10)$$

equation (2,9) is solvable, and the estimate

$$\|\Omega\| \leq \frac{|a| \cdot \|(E - K_1)^{-1}\| \cdot \|PT(\lambda)u\|}{1 - \|R + (\lambda - \lambda_1)S\|} \quad (2,11)$$

holds.

We now substitute (2,11) into (2,5), and (2,5) and (2,3) into (2,4). The inequality thus obtained, together with (2,10), determines a certain region in the  $\lambda$ -plane. Any closed contour  $C$  in this region that encloses the point  $\lambda_1$  will satisfy the conditions of Rouché's theorem and contain the unique eigenvalue of the operator (2,1).

In the particular case  $\|\tilde{K}(\lambda_1)\| < 1$ , instead of (2,8) one may put  $K_1 = 0$ , and then  $R = \tilde{K}(\lambda_1)$ ,  $S = \tilde{N}$ ,  $\|(E - K_1)^{-1}\| = 1$ .

3. We shall seek the contour  $C$  in the form of a circle of radius  $\rho$  with center at the point  $\lambda_1$ . Strengthening condition (2,10), we replace it by two inequalities

$$\|R\| < 1, \quad \rho < \frac{1 - \|R\|}{\|S\|}. \quad (3,1)$$

If  $\lambda \in C$ , then, obviously,

$$\|T(\lambda)A\| \leq \|T(\lambda_1)A\| + \rho\|NA\|, \quad \|PT(\lambda)u\| \leq \|PT(\lambda_1)u\| + \rho\|PNu\|. \quad (3,2)$$

We now substitute successively (3,2) into (2,11) and (2,5), and (2,5) and (2,3) into (2,4). We obtain for  $\rho$  the inequality

$$\alpha\rho^2 + \beta\rho + \gamma < 0, \quad (3,3)$$

where

$$\begin{aligned} \alpha &= \|(E - K_1)^{-1}\| \cdot \|NA\| \cdot \|PNu\| + |(Nu, v)| \cdot \|S\|, \\ \beta &= \|(E - K_1)^{-1}\| \cdot (\|NA\| \cdot \|PT(\lambda_1)u\| + \|T(\lambda_1)A\| \cdot \|PNu\|) \\ &\quad - |(Nu, v)|(1 - \|R\|), \\ \gamma &= \|(E - K_1)^{-1}\| \cdot \|T(\lambda_1)A\| \cdot \|PT(\lambda_1)u\|. \end{aligned} \quad (3,4)$$

If the vector  $u$  is sufficiently close to the exact eigenvector  $y$  and the direction of the vector  $v$  is close to the direction of the vector  $Nu$ , then  $\|T(\lambda_1)u\|$  and  $\|PNu\|$  are small, and from formulas (3,4) it follows that  $\beta^2 - 4\alpha\gamma > 0$ ,  $\beta < 0$ . Hence the solution of inequality (3,3) will have the form

$$0 < \rho_1 < \rho < \rho_2, \quad (3,5)$$

where  $\rho_1, \rho_2$  are the roots of the quadratic trinomial in (3,3). If  $\rho_1$  satisfies the second condition (3,1), then in the disk  $|\lambda - \lambda_1| < \rho_1$  there lies the unique eigenvalue, and there are no other eigenvalues in such a neighborhood

of the point  $\lambda_1$ , which is determined by the inequality

$$|\lambda - \lambda_1| < \min \left( \rho_2, \frac{1 - \|R\|}{\|S\|} \right). \quad (3,6)$$

From what has been said above it follows that the optimal choice of  $v$ , from the standpoint of the quality of estimate (3,5), will be  $v = Nu/\|Nu\|$ . In this case  $\|PNu\| = 0$ ,  $|(Nu, v)| = \|Nu\|$ , and the estimate is simplified and improved.

Incidentally, it is easy to show that, if  $u$  is given, then  $\|T(\lambda)u\|$  attains its minimum for  $\lambda = (u - Mu, Nu)/\|Nu\|^2$ , i.e., the indicated choice of  $v$  is also optimal from the standpoint of the minimum of the residual  $T(\lambda)u$ .

Of course, one may seek the contour  $C$  not in the form of a circle. The possibilities for improving the estimate connected with this depend on the particular realization of the Hilbert space and of the operator  $K(\lambda)$ .

Knowing an estimate for  $\|\omega\|$ , we easily estimate the accuracy of finding the approximate eigenvector by the formula

$$\frac{\|\varphi - au\|}{\|au\|} = \frac{\|\omega\|}{|a|},$$

since  $|a|$  enters as a factor in the expression for  $\|\omega\|$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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