ON SOME PROPERTIES OF ABSOLUTES

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Abstract

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MATHEMATICS

S. ILIADIS

ON SOME PROPERTIES OF ABSOLUTES

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For every space* R, in the papers ([2],[5]) an absolute $\widetilde{R}$ was constructed. This is a completely regular, extremally disconnected ([1]) space which is θ-continuously ([8]), irreducibly, and perfectly ([2]) mapped onto R. Among all regular extremally disconnected spaces that can be irreducibly, θ-continuously, and perfectly mapped onto R, it is unique. A natural question is: for which spaces R does there exist an extremally disconnected space that is continuously, irreducibly, and perfectly mapped onto R? In Sec. 1, considering this question, we present a new method for constructing the absolute $\widetilde{R}$ for an arbitrary space R. By the same method one can also construct certain other extremally disconnected spaces associated with the space R. One of them, denoted by $\widehat{R}$, first, can be continuously, irreducibly, bicompactly, and canonically closed** mapped onto R, and, second, can be compactified in any such extremally disconnected space that can be irreducibly, θ-continuously, and perfectly mapped onto R. If a space R is mapped onto a space R' by means of a continuous, irreducible, bicompact, and canonically closed mapping, then $\widehat{R}$ is compactified in $\widehat{R}'$. A necessary and sufficient condition is also given for the spaces $\widetilde{R}$ and $\widehat{R}$ to coincide.

In Sec. 2, generalizations of certain properties of compactness type are introduced, and the preservation of these generalized properties under certain kinds of mappings is proved, in particular under passage to absolutes.

In Sec. 3, a necessary and sufficient condition is given for a space possessing an extremally disconnected bicompact extension to be compactifiable in a bicompactum. In addition, it is shown—in contrast to the separable case ([7])—that, by removing from certain bicompacta a countable nowhere dense set, one can obtain spaces that are not compactifiable in bicompacta.

Sec. 1. Let R be an arbitrary space. To each of its points x we assign a certain space R_x, satisfying the condition: for every x there exists a continuous mapping f_x of the space R_x onto R such that on the set $R_x^- = f_x^{-1}(R \setminus x)$ it is a homeomorphism. Moreover, if $x \neq y$, then R_x and R_y have no common elements. It is clear that the sets $M_x = R_x \setminus R_x^- = f_x^{-1}x$ are nonempty and closed.

Let

$$\widehat{V} = \bigcup_{y \in f_x V \setminus y} M_y \cup (V \cap M_x)$$

for any set V open in R_x.

Finally, for any set U open in R, let U^* be the largest of all such sets V open in R_x (for each fixed x) that $f_x V \setminus x = U \setminus x$, and let

$$\widetilde{U} = \bigcup_{x \in U} (U^* \cap M_x).$$

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* By a space we mean a Hausdorff space.
** This means that the image of every canonically closed set is closed.

Consider on the set $M = \bigcup_{x \in R} M_x$ two topologies: by $\widetilde{R}\{R_x\}$ we denote the space obtained from the set M by taking as a base of open sets all sets of the form $\widetilde{U}$, and by $\widehat{R}\{R_x\}$ the space consisting of the same points, but taking as a base the system of all sets of the form $\widehat{V}$, where $V \subseteq R_x$ and $x \in R$.

Lemma. The space $\widehat{R}\{R_x\}$ is Hausdorff and is a compactification of the space $\widetilde{R}\{R_x\}$.

Indeed, the Hausdorff axiom is easily verified. It remains only to indicate the equality

$$\widetilde{U} = \bigcup_{x \in U} \widehat{U}^x.$$

Let now ι be an operator assigning to every space R a certain H-closed extension ιR of it. Fix a space R, and for each of its points x define a mapping

$$f_x^\iota : R_x \to R$$

as follows: if x is an isolated point, then f_x^ι is the identity mapping ($R_x = R$); if this is not so, then f_x^ι is the natural mapping* of the space $\iota(R \setminus x)$ onto ιR (we assume that it exists), considered on the subspace

$$R_x^\iota = (f_x^\iota)^{-1} R.$$

Consider now three systems $\{R_x^\beta\}$, $\{R_x^\sigma\}$, and $\{R_x^\tau\}$, where β, σ, τ are the symbols of the extensions, respectively, of Čech, Fomin, and Katětov (see ([3],[8])).

Theorem 1. Always

$$\widetilde{R}\{R_x^\tau\} = \widetilde{R}\{R_x^\sigma\} = \widetilde{R},$$

where $\widetilde{R}$ is the absolute of the space R.

Let us note that the spaces $\widehat{R}\{R_x^\beta\}$ and $\widetilde{R}\{R_x^\beta\}$ need not be absolutes. Let f be the natural mapping of the set $M = \bigcup M_x$ onto R:

$$f_x^{-1} = M_x,$$

and let

$$\widehat{R} = \widehat{R}\{R_x^\sigma\}.$$

Theorem 2. The mapping $f : \widehat{R} \to R$ is continuous, irreducible, bicompact, and canonically closed, and the space $\widehat{R}$ itself is extremally disconnected.

Theorem 3. The space $\widehat{R}$ is naturally compactified onto every extremally disconnected space R' that can be θ-continuously, irreducibly, and perfectly mapped onto R.

The space $\widehat{R}$ is, in the following sense, analogous to the absolute $\widetilde{R}$.

Theorem 4. If the space R can be continuously, irreducibly, bicompactly, and canonically closedly mapped onto a space R', then $\widehat{R}$ is naturally compactified onto $\widehat{R}'$.

To find a necessary and sufficient condition for the coincidence of the spaces R and $\widehat{R}$, call a point x of the space R **unattainable** if there is no closed nowhere dense set Φ such that $x \in \Phi \setminus x$.

Theorem 5. The spaces $\widetilde{R}$ and $\widehat{R}$ coincide if and only if R is regular and all its points are unattainable.

§ 2. We shall call a system of open sets of the space R a θ**-cover** if the closures of these sets cover all of R. We shall call the space R θ**-compact** if into every countable open cover of it one can inscribe a finite θ-cover; we shall call it θ-**paracompact** if into every open cover of it one can inscribe a locally finite θ-cover; we shall call it θ**-finalcompact** if into every open cover of it one can inscribe a countable θ-cover**.

* For example,

$$R_x^\iota = (R \setminus x) \cup M_x^\iota,$$

where

$$M^\iota = \overline{\Gamma \setminus x},$$

and Γ is an arbitrary neighborhood of the point x. Yu. M. Smirnov drew my attention to the possibility of this general construction.

** The terms "inscribe" and "locally finite" are understood in the usual sense.

Theorem 6. The space R is θ-paracompact, respectively θ-compact, if and only if its absolute $\widetilde{R}$ is θ-paracompact, respectively θ-compact; if the absolute $\widetilde{R}$ is finally compact, then the space R is θ-finally compact.

Corollary. The properties of θ-paracompactness and θ-compactness are preserved in both directions under θ-continuous, perfect, irreducible mappings*.

Theorem 7. Every regular (in particular, the absolute $\widetilde{R}$ of any space R) θ-paracompact space is paracompact**.

Corollary. The space R is θ-paracompact if and only if its absolute $\widetilde{R}$ is paracompact.

Theorem 8. The absolute $\widetilde{R}$ of a space R is perfectly zero-dimensional*** if and only if the space R is θ-paracompact.

§ 3. Lemma. Let bR be an extremely disconnected bicompactification, and let H be its countable discrete subset in the "relative" topology; then $bR = \beta H$.

Theorem 9. The closure of any infinite subset of an extremely disconnected bicompactum (in particular, every infinite extremely disconnected bicompactum) has cardinality not less than the cardinality $2\mathfrak{c}$; every open subset of an extremely disconnected bicompactum containing at least one nonisolated point has cardinality $\geq 2\mathfrak{c}$.

Theorem 10. Let the space R have such an extremely disconnected bicompact extension βR that the remainder $N = \beta R \setminus R$ has cardinality $< 2\mathfrak{c}$ (in particular, is countable); then the space R is not compactifiable**** into a bicompactum.

Theorem 11. Let the space R have an extremely disconnected bicompact extension βR; in order that it be compactifiable into some bicompactum, it is necessary and sufficient that there exist such a mapping g of the remainder $N = \beta R \setminus R$ into the space R that the bicompact partition of βR, consisting of the individual points of the set $\beta R \setminus (N \cup gN)$ and of sets of the form $g^{-1}x \cup x$, be closed and continuous*****.

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Moscow State University
named after M. V. Lomonosov

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* Under continuous perfect mappings (irreducibility is not needed), the properties of paracompactness and compactness are preserved in both directions.

** This assertion is equivalent to one assertion of Michael ([4]).

*** This means that into any of its open covers one can insert an open cover by pairwise disjoint sets. Earlier it was proved (6) only that paracompactness of a space implies perfect zero-dimensionality of its absolute.

**** If a countable nowhere dense nonclosed set is removed from a compactum, then the remaining space is compactifiable into some compactum (7).

***** That is, all elements of the given partition are closed sets.

Note: Figure translations are in progress. See original paper for figures.

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