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Abstract

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THE SYNTHESIS PROBLEM FOR A CLASS OF AUTOMATIC CONTROL SYSTEMS SUBJECT TO RANDOM DISTURBANCES

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A class of automatic control systems with parameters varying in time is considered. An example is a system consisting of an element with a fractional-rational transfer function, closed by feedback with a transfer function inversely proportional to a linear function of time. The input $x(t)$ and output $y(t)$ of the system are random functions. Such systems are described by a linear differential equation of Laplace type

$$\sum_{k=0}^{n}(\alpha_k + \beta_k t) y^{(k)}(t) = x(t)$$

(α_k and β_k are real constants), which, for convenience in the subsequent exposition, we shall write in the form

$$Ly = [tQ(p) + Q'(p) - P(p)]y(t) = x(t), \tag{1}$$

where

$$Q(p) = \sum_{k=0}^{n} a_k p^k \ (a_n = 1); \qquad P(p) = \sum_{k=0}^{n-1} b_k p^k;$$

p is the differentiation operator; $\tau = t - t_0,\ t_0 > 0$. The system operates for $0 \le t < t_0$. Initial conditions: $y^{(k)}(0) = 0,\ k = 0, 1, \dots, n-1$.

The problem of synthesizing the system consists in choosing the system parameters so that the output signal $y(t)$ is as close as possible to the desired signal $\omega(t)$, which in general is some transformation of $x(t)$. According to the mean-square-error criterion, at the instant $t = T \in [0, t_0]$ one must minimize the functional

$$D(T) = E|y(T) - \omega(T)|^2,$$

where E is the symbol of mathematical expectation (it is impossible to minimize $D(T)$ for all $t \in [0, t_0]$, owing to the nonstationarity of the problem). To minimize $D(T)$ we have at our disposal the $2n$ parameters of equation (1), a_k and b_k, $k = 0, 1, \dots, n-1$. The order n of the equation is assumed fixed.

For automatic control systems described by equations with constant coefficients, this problem was considered by Phillips ([1]). In the present note his method is extended to systems described by equation (1).

1. The determination of the best a_k, b_k is based on the use of the explicit expression for the solution of equation (1) with zero initial data, obtained in ([2]):

$$y(t) = \int_0^t A(u,t) x(u)\, du, \tag{2}$$

where the weighting function is

$$A(u,t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{z(t_0 - u)}}{Q(z)} \int_{+\infty}^{z} e^{\zeta \tau} \exp\left[- \int_z^{\zeta} \frac{P(\eta)}{Q(\eta)}\, d\eta \right] d\zeta\, dz;$$

γ such that all roots of the polynomial $Q(z)$ lie to the left of the contour of integration.

Using (2), we represent the functional $D(t)$ in the form

$$D(T) = \int_0^T \int_0^T A(u,T) A(v,T) R(u,v)\, du\, dv - 2 \int_0^T A(u,T) S(u,T)\, du + E\left[|w(T)|^2 \right], \tag{3}$$

where $R(u,v) = E[x(u), x(v)]$ is the correlation function of the signal $x(t)$; $S(u,T) = E[x(u), w(T)]$ is the cross-correlation function of $x(t)$ and $w(t)$.

Let, for certain polynomials $P(p)$ and $Q(p)$, the functional $D(T)$ be minimal. We vary the coefficients of the polynomials $Q(p)$ and $P(p)$. Denote

$$Q_1(p) = \sum_{k=0}^{n-1} p^k \delta a_k, \qquad P_1(p) = \sum_{k=0}^{n-1} p^k \delta b_k,$$

where δa_k and δb_k are variations of the coefficients. Then the first variation of $A(u,t)$ is

$$A_1(u,t) = -\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{z(t_0 - u)}}{Q(z)} \left[\int_{+\infty}^{z} e^{\zeta \tau} \exp\left[- \int_z^{\zeta} \frac{P(\eta)}{Q(\eta)}\, d\eta \right] \frac{Q_1(z)}{Q(z)} \right.$$

$$+\int_{+\infty}^{z} e^{\zeta\tau} \exp\left[-\int_{z}^{\zeta} \frac{P(\eta)}{Q(\eta)}\, d\eta\right] \int_{z}^{\zeta} \frac{P(\eta)}{Q(\eta)} \left(\frac{P_1(\eta)}{P(\eta)} - \frac{Q_1(\eta)}{Q(\eta)}\right) d\eta \Bigg]\, d\zeta\, dz, \quad (4)$$

and the second variation is

$$A_2(u,t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{z(t_0 - u)}}{Q(z)} \int_{+\infty}^{z} e^{\zeta\tau} \exp\left[-\int_{z}^{\zeta} \frac{P(\eta)}{Q(\eta)}\, d\eta\right] \times$$

$$\times \left\{ \frac{1}{2} \left[\int_{z}^{\zeta} \frac{P(\eta)}{Q(\eta)} \left(\frac{P_1(\eta)}{P(\eta)} - \frac{Q_1(\eta)}{Q(\eta)}\right) d\eta\right]^2 - \int_{z}^{\zeta} \frac{P(\eta)}{Q(\eta)} \left(\frac{Q_1^2(\eta)}{Q^2(\eta)} - \frac{P_1(\eta)}{P(\eta)} \frac{Q_1(\eta)}{Q(\eta)}\right) d\eta\right.$$

$$\left. + \frac{Q_1(z)}{Q(z)} \int_{z}^{\zeta} \frac{P(\eta)}{Q(\eta)} \left(\frac{P_1(\eta)}{P(\eta)} - \frac{Q_1(\eta)}{Q(\eta)}\right) d\eta + \frac{Q_1^2(z)}{Q^2(z)} \right\} d\zeta\, dz.$$

Using (3), one can find $D_1(T)$ and $D_2(T)$—the first and second variations of $D(T)$:

$$D_1(T) = 2\int_0^T A_1(u,T) \left[\int_0^T R(u,v) A(v,T)\, dv - S(u,T)\right] du,$$

$$D_2(T) = \int_0^T \int_0^T [A(u,T)A_2(v,T) + A_1(u,T)A_1(v,T) +$$

$$+ A_2(u,T)A(v,T)]\, R(u,v)\, du\, dv - \int_0^T A_2(u,T) S(u,T)\, du.$$

If no restrictions are imposed on the coefficients, then a necessary condition for the minimum of $D(T)$ is the equality to zero of $D_1(T)$:

$$\int_0^T A_1(u,T) \left[\int_0^T R(u,v) A(v,T)\, dv - S(u,T)\right] du = 0. \quad (5)$$

for arbitrary δa_k and δb_k. Therefore, taking into account the linear dependence of $A_1(u,T)$ on δa_k and δb_k (which follows directly from (4)), we obtain, as a necessity, that all coefficients of δa_k and δb_k in (5) must be equal to zero. This gives $2n$ transcendental equations for determining the $2n$ unknowns a_k and b_k. If some of the a_k, b_k are fixed, the corresponding δa_k and δb_k in (5) are set equal to zero. A sufficient condition for a local minimum of $D(T)$ is the positivity of the second variation of $D(T)$ for all δa_k and δb_k not simultaneously equal to zero.

2. Let us consider several simple examples illustrating the peculiarities that arise in such problems.

Example 1. The system is described by the equation:

$$\tau(y' + ay) + y = x.$$

The input signal $x(t)$ is assumed to be white noise, i.e. a stationary random process with $R(u,v) = \delta(u-v)$. It is required to choose the parameter a so that $y(t)$ be close at the moment $t = T$ to

$$w(t) = -\int_0^t x(\eta)\, d\eta$$

in the sense of minimizing $D(T)$. The necessary condition for a minimum of $D(T)$ is written as follows:

$$\int_0^T (T-u)e^{-a(T-u)} \left[\int_0^T \delta(u-v) \frac{e^{-a(T-v)}}{\tau}\, dv + 1 \right] du = 0, \qquad \tau = T - t_0.$$

After integration we obtain

$$1 + \frac{1}{4\tau} - e^{-2aT}\frac{2aT+1}{4\tau} - e^{-aT}(aT+1) = 0.$$

Investigating this equation, one can establish that there exists a unique solution a for $\tau < -1/4$, and as $\tau \to -1/4$ from the left $a(\tau)$ has the asymptotic form

$$a(\tau) = \frac{1}{T}\left[\ln \frac{4\tau}{1+4\tau} + \ln\ln \frac{4\tau}{1+4\tau} + o(1)\right],$$

i.e. $a(\tau) \to \infty$ as $\tau \to -1/4$ from the left according to a logarithmic law. For $-1/4 \le \tau < 0$, $a = \infty$, i.e. an optimal system does not exist.

Thus, in the present case the existence of an optimal system depends on the moment T at which the functional $D(T)$ is minimized. However, under certain restrictions on $x(t)$ and $w(t)$, one can guarantee the existence of an optimal system. Namely, the following is true.

Theorem. If $S(u,T)$ and $R(u,v)$ are continuous for $0 \le u, v \le T$ and $S(T,T) < 0$, then for the system described by the equation

$$\tau(y' + ay) + y = x,$$

a solution of the optimal problem always exists.

Example 2. The system is described by the equation:

$$\tau(y'' + ay) + 2y' = x.$$

Again $x(t)$ is white noise and

$$w(t) = -\int_0^t x(\eta)\,d\eta.$$

The condition that $D_1(T)$ be equal to zero is:

$$\int_0^T \left[(T-u)\cos\sqrt{a}(T-u) - \frac{\sin\sqrt{a}(T-u)}{\sqrt{a}}\right]\left[\int_0^T \frac{\sin\sqrt{a}(T-v)}{\sqrt{a}}\delta(u-v)\,dv + 1\right] du = 0.$$

Hence one obtains the relation for determining

$$\sin\sqrt{a}T = f(a), \qquad \text{where } f(a) = O\left(\frac{1}{\sqrt{a}}\right). \qquad (a \to \infty),$$

from which it follows that the necessary minimum condition is satisfied by an infinite set of values of a.

Remark 1. If restrictions are imposed on the coefficients of the equation a_k and b_k, arising, for example, from considerations of stability of the solution, then the minimum of $D(T)$ may also be attained on the boundary, where condition (5) need not necessarily be satisfied.

Remark 2. In practical applications the coefficients a_k and b_k depend on a small number of parameters λ_j, $j = 1, \dots, l$ $(l = 3 \div 7)$.

In this case, in condition (5) one should replace δa_k and δb_k by

$$\sum_{j=1}^{l} \frac{\partial a_k}{\partial \lambda_j}\,\delta\lambda_j$$

and

$$\sum_{j=1}^{l} \frac{\partial b_k}{\partial \lambda_j}\,\delta\lambda_j,$$

respectively, and set equal to zero the coefficients of $\delta\lambda_j$. As a result, the number of equations for determining the l unknowns λ_j turns out to be equal to l.

Remark 3. Suppose that, before entering the automatic control system described at the beginning of the note, the signal $x(t)$ is transformed by the differential operator

$$M = \sum_{k=0}^{m} (c_k + d_k t) \frac{d^k}{dt^k} \qquad (m < n),$$

i.e., the output signal $y(t)$ is related to the input $x(t)$ by the relation $Ly = Mx$. It is necessary to choose not only the best a_k and b_k entering into the operator L, but also c_k and d_k. This problem fits into the scheme described above, since it can be shown that

$$y(t) = \int_0^t B(u,t) x(u)\, du,$$

where

$$B(u,t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{z(t_0 - u)}}{Q(z)} [T(z) + uV(z)] \int_{+\infty}^{z} e^{\zeta\tau} \exp\left[-\int_z^{\zeta} \frac{P(\eta)}{Q(\eta)}\, d\eta \right] d\zeta\, dz,$$

$$T(z) = \left(\sum_{k=0}^{m} c_k z^k - \sum_{k=1}^{m} k d_k z^{k-1} \right), \qquad V(z) = \sum_{k=0}^{m} d_k z^k,$$

and therefore the first and second variations of $D(T)$ can be found by the same method.

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Note: Figure translations are in progress. See original paper for figures.