On Order-Extendable Groups

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Abstract

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On Order-Extendable Groups

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In this note one finds subgroups of an order-extendable group, definable independently of the order, by which the factor-groups are order-extendable. In particular, these subgroups are of interest in connection with questions of describing the ways of ordering a group. Next, the connection is indicated between the problem of the coincidence of the classes of order-extendable and orderable groups and other questions.

A group is called **order-extendable** if every partial order on it can be extended to a linear one.

Notation: $S(a, b, ...)$ is the minimal invariant semigroup in the group G containing $a, b, ...$; $\overline{M}$ is the image of the set M from the group G under a homomorphism of G onto the group $\overline{G}$.

A subgroup H of a group G is called **strictly isolated** if from

$$xg_1^{-1}xg_1 \dots g_n^{-1}xg_n \in H$$

it follows that

$$x,\ g_1^{-1}xg_1, \dots,\ g_n^{-1}xg_n \in H.$$

Subgroups and subsemi-groups of this kind were considered in ([1],[2]).

Theorem 1. *The factor-group G/H of an order-extendable group G is order-extendable if and only if H is an invariant strictly isolated subgroup.*

The necessity follows from the strict isolation of the identity in an ordered group. To prove sufficiency, suppose at first that H is strictly isolated and $G/H = \overline{G}$ is not orderable. Then, on the basis of Theorem 1 from ([1]), in $\overline{G}$ there are elements $\overline{a}_1, \dots, \overline{a}_n$, distinct from $\overline{1}$, such that for arbitrary $\varepsilon_i = \pm 1$

$$S(\overline{a}_1^{\varepsilon_1}, \dots, \overline{a}_n^{\varepsilon_n}) \cap \overline{1} \neq \phi^*$$

and, consequently, in G also

$$S(a_1^{\varepsilon_1}, \dots, a_n^{\varepsilon_n}) \cap H \neq \phi. \quad (*)$$

Here one may assume n minimal, i.e., no proper part of $a_1, \dots, a_n$ has this property. Let, further, P be the semigroup of positive elements (without the identity) from H under some ordering of G, and

$$P_1 = S(a_1)H \cup P.$$

P_1 is an invariant semigroup by virtue of the invariance of H, $S(a_1)$, and P. We shall show that P_1 does not contain the identity. If $s \in S(a_1)$, $h \in H$, and $sh = 1$, then $s \in H$, and, by the strict isolation of H, it follows that $a_1 \in H$, which contradicts $\overline{a}_1 \neq \overline{1}$. Thus P_1 is an invariant semigroup not containing the identity and, hence, it defines in G a partial order. Starting from P_1, extend the order of G. We shall assume $a_1, \dots, a_n > 1$, since if $a_i < 1$, one may take a_i^{-1}. Moreover, for every $h \in H$ we shall have $a_1 \gg h$ (i.e. $a_1 > h^k$ for every integer k), since $a_1 h^{-k} \in S(a_1)H \subset P_1$. Next take the intersection H_1 of all convex subgroups containing H. Let

$$a_1, \dots, a_m \in H_1, \quad a_{m+1}, \dots, a_n \notin H_1.$$

$m \geq 1$, since from $a_1 \in H$ it follows that $a_1 \in H_1$. On the basis of convexity and invariance of H_1, for any $g \in G$, $h \in H_1$ we shall have

$$g^{-1} a_1 g, \dots, g^{-1} a_{mg} \gg h,$$

and hence

$$g^{-1} a_1 g, \dots, g^{-1} a_{mg} \in H_1.$$

Therefore every product composed of elements of H_1 and at least one of

$$g^{-1} a_1 g, \dots, g^{-1} a_{mg}$$

also does not belong to H_1, and hence not to H.

$*$ ϕ is the empty set.

Consequently,

$$S(a_1, \dots, a_n) \cap H = S(a_{m+1}, \dots, a_n) \cap H \neq \emptyset.$$

But, by the minimality of n, there exist such $\varepsilon_{m+1}, \dots, \varepsilon_n$ that

$$S(a_{m+1}^{\varepsilon_{m+1}}, \dots, a_n^{\varepsilon_n}) \cap H \neq \emptyset,$$

and, therefore,

$$S(a_1, \dots, a_m, a_{m+1}^{\varepsilon_{m+1}}, \dots, a_n^{\varepsilon_n}) \cap H = \emptyset,$$

which contradicts $(*)$. Hence G/H is orderable.

We now show that $G/H = \overline{G}$ is an orderable group. Let $\overline{P}_2$ be the semigroup of positive elements of $\overline{G}$ under some partial ordering of $\overline{G}$, and let P_2 be its complete preimage in G. Starting from the partial order defined by P_2, we order G. In this case, from $P_2 = P_2 H$ it follows that $P_2 \cap H_2 = \emptyset$, where H_2 is the intersection of all convex subgroups containing H. On the basis of ([3]), from the orderability of G/H_2 and the possibility of ordering H_2/H, which is preserved under the action of inner automorphisms from G/H, the group $\overline{G}$ can be ordered while preserving the orders on G/H_2 and on H_2/H. This order will be an extension of $\overline{P}_2$, since $P_2 \cap H = \emptyset$.

Corollary. If in an orderable group G a subgroup H is invariant and strictly isolated, then there exists an ordering of G under which H is convex.

Lemma. If in an ordered group G: 1) an element z belongs to the center of the group G; 2) A is the union of all convex subgroups not containing z; 3) $xg_1^{-1}xg_1 \dots g_n^{-1}xg_n = z$, then $\overline{x}^{n+1} = \overline{z}$ in $\overline{G} = G/A$.

Let B be the intersection of all convex subgroups containing z. From 1), 2), and the construction of B it follows that A and B are invariant. By convexity and invariance of A and B we have $x, g_1^{-1}xg_1, \dots, g_n^{-1}xg_n \in B \setminus A$. The inner automorphisms of the group G induce in B/A automorphisms that are order-preserving (see ([4])). Therefore, for any $h \in G$ we have

$$\overline{z} = \overline{h^{-1}zh} = \overline{h^{-1}xh \cdot h^{-1}g_1^{-1}xg_1h \dots h^{-1}g_n^{-1}xg_{nh}}$$

$$= \overline{(h^{-1}xh) \cdot g_1^{-1}(h^{-1}xh)g_1 \dots g_n^{-1}(h^{-1}xh)g_n}.$$

Suppose, for definiteness, that $\overline{x} > \overline{h^{-1}xh}$ in $\overline{G}$. But then

$$\overline{xg_1^{-1}xg_1 \dots g_n^{-1}xg_n} >$$

$$> \overline{(h^{-1}xh)g_1^{-1}(h^{-1}xh)g_1 \dots g_n^{-1}(h^{-1}xh)}.$$

Consequently, $\overline{x} = \overline{h^{-1}xh}$, and therefore $\overline{x}^{n+1} = \overline{z}$ in $\overline{G}$.

Theorem 2. The quotient group G/Z of an orderable group G by its center Z is orderable.

For the proof, by Theorem 1, it is enough to show that the center Z is strictly isolated. Order G, and let $z \in Z$; $z \in B \setminus A$, where A and B are neighboring convex subgroups; P is the set of positive elements of G not belonging to $B \setminus A$; $I^+(z)$ is the set of positive elements of $I(z)$, where $I(z)$ is the isolator (see ([5])) of z in G. By the isomorphism of $I(z)$ to an additive group of rational numbers and by the Archimedean property of B/A, it follows that $I(z)\setminus 1 \subseteq B\setminus A$. From this, convexity and invariance of A and B imply that

$$P_1 = PI(z) \cup I^+(z)$$

is an invariant semigroup not containing the identity, and hence P_1 defines a partial order in G. Starting from P_1, order G. Let under this ordering H be the intersection of all convex subgroups containing z. Then $H \setminus 1 \subseteq B \setminus A$, since, if $g \in B \setminus A$,

then either $g \in P$, or $g \in P^{-1}$. We may further assume that the order in the subgroup H coincides with its order under the original ordering of G. From this, $H \setminus 1 \subseteq B \setminus A$, and the Archimedeanness of B/A, it follows that H is Archimedean and is the least convex subgroup distinct from the identity under the new ordering of G. Therefore, on the basis of the lemma, the equality

$$xg_1^{-1}xg_1 \cdots g_n^{-1}xg_n = z$$

implies $x^{n+1} = z$ in $G/\{1\} = G$. From $x^{n+1} = z$ and the isolatedness of $I(z)$ it follows that $x \in I(z) \subseteq Z$, and hence $g_1^{-1}xg_1, \dots, g_n^{-1}xg_n \in Z$. The theorem is proved.

Remark 1. From the isomorphism of an ordered group to a factor group of some free ordered group (see ([4])), Theorems 1 and 2 imply the equivalence of the following propositions.

1) An orderable group is bi-orderable.
2) A free group is bi-orderable.
3) A group is bi-orderable if and only if its identity is strictly isolated.
4) The factor group of a group with strictly isolated identity by the center is bi-orderable.

Remark 2. A two-step solvable group is orderable if and only if its identity is strictly isolated.

This is proved by a direct application of Theorem 3 from ([1]) to the system consisting of the group itself, the isolator of its commutant, and the identity.

Remark 3. The factor group G/Z_1 of an orderable group G by a complete subgroup Z_1 of the center Z is orderable.

For the proof, take the system of all convex subgroups under some ordering of G:

$$G \supset \cdots \supset B \supset A \supset \cdots \supset C \supset \cdots \supset 1 \qquad (**)$$

and multiply each subgroup of this system by Z_1. Then, together with the identity, we obtain the system

$$G \supset \cdots \supset Z_1B \supset Z_1A \supset \cdots \supset Z_1C \supset \cdots \supset Z_1 \supset 1, \qquad (***)$$

which, as we shall show, satisfies the conditions of Theorem 3 from ([1]). This will prove the assertion, since under the order constructed in ([1]), the subgroups of the system $(***)$ will be convex. We shall show the validity of only one of these conditions (the strict isolatedness of the subgroups of the system $(***)$), since the fulfillment of the remaining ones follows directly from their fulfillment for the system $(**)$.

Let

$$xg_1^{-1}xg_1 \cdots g_n^{-1}xg_n = zc \in Z_1C,$$

where $z \in Z_1$, $c \in C$, $z \in C$ and $z \in B \setminus A$, where A and B are adjacent convex subgroups of G. Then, by the lemma, in G/A we have $\overline{z} = \overline{x}^{n+1}$, and hence $z = x^{n+1}a$, where $a \in A$. By the completeness of Z_1,

$$(z_1x^{-1})^{n+1} = a,$$

where $z_1^{n+1} = z$ and $z_1 \in Z_1$. From the strict isolatedness of A, $(z_1x^{-1})^{n+1} = a$ implies $z_1x = a_1^{-1} \in A$, i.e. $x = z_1a_1$.

Further, from

$$zc = (z_1a_1)g_1^{-1}(z_1a_1)g_1 \cdots g_n^{-1}(z_1a_1)g_n$$

it follows that

$$a_1g_1^{-1}a_1g_1 \cdots g_n^{-1}a_1g_n = c \in C,$$

whence, on the basis of the strict isolatedness of C,

$$a_1,\ g_1^{-1}a_1g_1, \dots,\ g_n^{-1}a_1g_n \in C,$$

and therefore $x = z_1a_1 \in Z_1C$. Consequently,

$$x,\ g_1^{-1}xg_1, \dots,\ g_n^{-1}xg_n \in Z_1C.$$

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Note: Figure translations are in progress. See original paper for figures.