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Abstract

Full Text

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THE SEMIGROUP OF VARIETIES OF GROUPS

(Presented by Academician P. S. Aleksandrov, 18 X 1962)

A variety is any class $\mathfrak{M}$ of groups closed under the operations of taking subgroups, homomorphic images, and complete direct products of groups from $\mathfrak{M}$. On the other hand (see ([1])), a variety is the class of all groups satisfying some system of identities; therefore to each variety there corresponds a certain verbal subgroup (see ([2])).

H. Neumann ([3]) introduced an associative operation on the set of all varieties: the **product** $\mathfrak{A}\mathfrak{B}$ of varieties $\mathfrak{A}$ and $\mathfrak{B}$ is the variety consisting of all possible extensions of groups from $\mathfrak{A}$ by means of groups from $\mathfrak{B}$. Moreover, if the varieties $\mathfrak{A}$ and $\mathfrak{B}$ correspond to the verbal subgroups U and V, then to their product $\mathfrak{A}\mathfrak{B}$ there corresponds the verbal subgroup $U(V)$, generated by all possible words of the form $u(v_1, \dots, v_n)$ with $u(x_1, \dots, x_n) \in U$, $v_i \in V_i$, $i = 1, \dots, n$.

In what follows, by the **semigroup** of varieties we shall mean the set of all varieties, except the variety of all groups and the variety consisting of the one-element group, with the operation defined above.

In ([3]) it was proved that the right cancellation law holds in the semigroup of varieties, and that the indecomposable (with respect to product) varieties are generators of this semigroup. In the same work H. Neumann posed the question whether the semigroup of varieties is free. The present note is devoted to a positive solution of this problem.*

The main tool for solving the problem will be the operations of the discrete and complete **wreath product** of groups. The discrete (complete) wreath product $A \operatorname{wr} B$ ($A \operatorname{Wr} B$) is the split extension of the direct product

$$\bar{A} = \prod_{b \in B}^{\times} A(b)$$

(the complete direct product

$$\tilde{A} = \prod_{b \in B}^{\sim\times} A(b)$$

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) by means of the group B, where $A(b) \cong A$, $b \in B$, such that $b_1^{-1} a(b) b_1 = a(bb_1)$; $b, b_1 \in B$.

Lemma 1. *For any extension G of a group A by means of a group B there exists a monomorphism $\varphi : G \to A \operatorname{Wr} B$, and the commutative diagram*

$$\begin{array}{ccccccccc} E & \to & A & \to & G & \to & B & \to & E \\ & & \downarrow \varphi_A & & \downarrow \varphi & & \downarrow \varepsilon_B & & \\ E & \to & \tilde{A} & \to & A \operatorname{Wr} B & \to & B & \to & E, \end{array}$$

holds, where ε_B is the identity automorphism of the group B; φ_A is the restriction of φ to the subgroup A.

Proof (cf. ([4,5])). In each coset of G modulo A we choose a representative s_b. Then φ is defined as follows:

$$g_1 \varphi = b_1 \prod_{b \in B} s_{bb_1}^{-1} g_1 s_b(b),$$

where b_1 is the image of g_1 in B. The lemma is verified without difficulty.

Lemma 2. *All identities of the group $A \operatorname{wr} B$ also hold in the group $A \operatorname{Wr} B$.*

Proof. Indeed, suppose that an identity $v(x_1, \dots, x_n)$ of the group $A \operatorname{wr} B$ fails for some $g_1, \dots, g_n \in A \operatorname{Wr} B$, i.e. $v = v(g_1, \dots, g_n) \neq 1$. Obviously, $v \in \tilde{A}$; hence some, say the b_0-th, component of the element v is not equal to 1, but in the process of reducing the word $v(g_1, \dots, g_n)$ to canonical form in the sense of the group $A \operatorname{Wr} B$, in forming each component only

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* *Note added in proof.* After the manuscript had been submitted for publication, the author learned of the work ([6]), in which the problem is also solved.

finite number of components of the elements $g_1, \dots, g_n$. In that case, if the elements g_i are replaced by $g_i' \in A \operatorname{wr} B \subseteq A \operatorname{Wr} B$, obtained from g_i by replacing by 1 all components except the component from B and also those which take part in the formation of the b_0-th component of the element v, then we obtain that

$$v(g_1', \dots, g_n') \neq 1,$$

since this element has the same b_0-th component as v. The contradiction obtained proves the lemma.

Thus, now, in order to prove that any extension of A by means of B belongs to some multivariety, it suffices, as follows from Lemmas 1 and 2, to prove that $A \operatorname{wr} B$ lies in this multivariety.

Lemma 3. *In the semigroup of multivarieties the law of left cancellation holds, i.e. from $\mathfrak{A}\mathfrak{B}_1 = \mathfrak{A}\mathfrak{B}_2$ it follows that $\mathfrak{B}_1 = \mathfrak{B}_2$.*

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Proof. Let the verbal subgroups U, V_1, V_2 correspond to the multivarieties $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$. Take the group

$$F = A \operatorname{wr} B,$$

where A is the $\mathfrak{A}$-free group of countable rank with free generators $a_1, a_2, \ldots$,

$$B = \prod_{i=0}^{\infty}{}^* B_i,$$

and each group B_i is a $\mathfrak{B}_1$-free group of countable rank with free generators $b_{i1}, b_{i2}, \ldots$. Take $V_2(F)$ ($V_2(F) \in \mathfrak{A}$, since $U(V_2(F)) = U(V_1(F)) = E$), and prove that $V_2(F) \subseteq \bar{A}$. Suppose, on the contrary, that there is an element $b \cdot \bar{a} \in V_2(F)$, $b \neq 1$. Suppose, for definiteness, that the projection b_0 of the element b onto B_0 is not equal to 1. It has the form $b_0 = f(b_{01}, \ldots, b_{0n})$; obviously, $b_0 \in V_2(F)$ and in $V_2(F)$ there are also contained the nonidentity elements

$$b_i = f(b_{i1}, \ldots, b_{in}).$$

The commutators contained in $V_2(F)$,

$$k_i = [b_0, a_i(1)] = a_i(1) \cdot a_i^{-1}(b_0),$$

obviously freely generate an $\mathfrak{A}$-free group

$$K = \{k_1, k_2, \ldots\}.$$

Take in $V_2(F)$ the subgroup $\{K, B'\}$, where

$$B' = \{b_1, b_2, \ldots\} = \prod_{i=1}^{\infty}{}^* \{b_i\},$$

and prove that

$$\{K, B'\} \cong K \operatorname{wr} B'.$$

Indeed, denote by

$$K(1) = K, \qquad K(b') = b'^{-1} K b', \quad b' \in B'.$$

It is checked directly that

$$\{K(b'),\, b' \in B'\} = \prod_{b' \in B'}{}^* K(b'),$$

and if $k(b') \in K(b')$, then

$$b''^{-1} k(b') b'' = k(b' b'')$$

for $b'' \in B'$, i.e.

$$\{K, B'\} \cong K \operatorname{wr} B'.$$

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Thus $K \operatorname{wr} B' \in \mathfrak{A}$, since this group is embedded in $V_2(F) \in \mathfrak{A}$. Hence, according to Lemmas 1 and 2, $\mathfrak{A}$ contains every extension of K by means of B'. But K is a free group of countable rank of the multivariety $\mathfrak{A}$, and B' is also a free group of some nonunit multivariety $\mathfrak{B}$ of abelian groups; therefore $\mathfrak{A}$ contains the free group of countable rank of the multivariety $\mathfrak{A}\mathfrak{B}$. Therefore $\mathfrak{A} = \mathfrak{A}\mathfrak{B}$, but then $\mathfrak{A} = \mathfrak{A}\mathfrak{B}^s$ for any s. This is impossible, since (see (3)) the number of factors in a decomposition of the multivariety $\mathfrak{A}$ cannot exceed the minimal length of the elements of U. The lemma is proved.

Lemma 4. *If* $\mathfrak{A}_1\mathfrak{B}_1 = \mathfrak{A}_2\mathfrak{B}_2$, *then either* $\mathfrak{A}_1 \supseteq \mathfrak{A}_2$, $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$, *or* $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$, $\mathfrak{B}_1 \supseteq \mathfrak{B}_2$.

Proof. Denote by U_1, U_2, V_1, V_2 the verbal subgroups corresponding to the multivarieties $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_1, \mathfrak{B}_2$, respectively. Suppose $\mathfrak{B}_1 \not\subseteq \mathfrak{B}_2$. Take the group

$$F = A \operatorname{wr} B,$$

where A is the $\mathfrak{A}_1$-free group of countable rank with free generators $a_1, a_2, \ldots$, and B is the $\mathfrak{B}_1$-free group of countable rank. By assumption, $V_2(B) \neq E$; let $b \in V_2(B)$, $b \neq 1$. Then $V_2(F)$ contains the elements

$$k_i = [b, a_i(1)] = a_i(1)a_i^{-1}(b),$$

freely generating an $\mathfrak{A}_1$-free group K. Since

$$U_2(V_2(F)) = E,$$

we have

$$U_2(K) = E$$

and $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$. If, in addition, $\mathfrak{B}_2 \not\subseteq \mathfrak{B}_1$, then in exactly the same way we show that $\mathfrak{A}_2 \subseteq \mathfrak{A}_1$ and, using the law of left cancellation, find $\mathfrak{B}_1 = \mathfrak{B}_2$. Therefore $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$, as was required. There remains the case $\mathfrak{B}_1 = \mathfrak{B}_2$, but here we may use H. Neumann' s law of right cancellation, already proved, and obtain $\mathfrak{A}_1 = \mathfrak{A}_2$. The lemma is proved.

Theorem. *The semigroup of multivarieties is a free semigroup with the indecomposable multivarieties as the system of free generators.*

Proof. Since H. Neumann showed that every

a variety is represented as a product of indecomposable ones, it remains to prove the uniqueness of such a representation. Let $\mathfrak{P} = \mathfrak{A}_1\mathfrak{B}_1 = \mathfrak{A}_2\mathfrak{B}_2$, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are indecomposable. We shall prove that $\mathfrak{A}_1 = \mathfrak{A}_2$. Suppose that $\mathfrak{A}_1 \neq \mathfrak{A}_2$; then also $\mathfrak{B}_1 \neq \mathfrak{B}_2$. By Lemma 4 the factors are connected by certain inclusions; let, for example, $\mathfrak{A}_1 \subset \mathfrak{A}_2$, $\mathfrak{B}_1 \supset \mathfrak{B}_2$. Take the group $F = A \operatorname{Wr} B$, where A is an $\mathfrak{A}_1$-free group and B is a $\mathfrak{B}_1$-free group of countable rank. By Lemma 1, F contains a $\mathfrak{P}$-free group Φ of countable rank. The group $V_2(\Phi)$ is an $\mathfrak{A}_2$-free group of countable rank, and $V_2(\Phi) \supset V_1(\Phi)$ (since $\mathfrak{B}_1 \supset \mathfrak{B}_2$); therefore $V_2(\Phi)$ is an extension of the group $V_1(\Phi) \in \mathfrak{A}_1$ by means of $V_2(\Phi)/V_1(\Phi) \cong V_2(B) \neq E$

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(the isomorphism follows from Lemma 1); denote $V_2(B) = G$. Let $\mathfrak{M} = \mathfrak{M}(G)$ be the variety generated by the group G, i.e. the class of all groups in which all identical relations holding in G are satisfied. We shall show that any $\mathfrak{M}$-free group is embedded in the complete direct product of a certain number of copies of the group G. Indeed, let X be a set of arbitrary cardinality. Take the set of all single-valued mappings $\varphi_i : X \to G$, $i \in I$, and the group

$$S = \prod_{i \in I}^{\sim X} G_i, \qquad G_i \cong G.$$

Then, if by φ we denote the mapping

$$x\varphi = \prod_{i \in I} x\varphi_i \in S, \qquad x \in X,$$

then, obviously, φ is one-to-one, and $X\varphi$ freely generates an $\mathfrak{M}$-free group. Now take the group $F' = A' \operatorname{wr} B'$, where A' is an $\mathfrak{A}_1$-free group of arbitrary rank with free generators a_j, $j \in J$,

$$B' = B \times \prod_{i \in I}^{\sim X} B_i, \qquad B_i \cong B.$$

Obviously,

$$V_2(B') = V_2(B) \times \prod_{i \in I}^{\sim X} V_2(B_i).$$

We may assume that $V_2(B_i) = G_i$. Take in $G = V_2(B)$ an element $g \neq 1$. In $V_2(F')$ lie the commutators $k_j = [g, a_j(1)]$, $j \in J$, which freely generate a group $K \cong A'$. Just as in the proof of Lemma 3, it is easy to show that

$$\{K, S\} \cong K \operatorname{wr} S, \qquad \text{where } S = \prod_{i \in I}^{\sim X} G_i.$$

Take in S an $\mathfrak{M}$-free subgroup M, generated by the set $X\varphi$; obviously, $\{K, M\} \cong K \operatorname{wr} M$. The latter group is embedded in $V_2(F') \in \mathfrak{A}_2$, and hence itself lies in $\mathfrak{A}_2$. Further, since for any normal divisor N of the group K one has

$$(K \operatorname{wr} M)/\bar{N} \cong (K/N) \operatorname{wr} M,$$

where $\bar{N}$ is the normal divisor generated by N in $K \operatorname{wr} M$, it follows, by the closure of varieties with respect to homomorphisms, that (using Lemmas 1 and 2) $\mathfrak{A}_2$ contains all extensions of any group $H \in \mathfrak{A}_1$ by means of any $\mathfrak{M}$-free group M. Let $R = M\psi$ be an arbitrary group from $\mathfrak{M}$. Take a split extension Γ of the group

$$\bar{H} = \prod_{r \in R}^{\sim X} H(r)$$

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by means of M, where $H(r) \cong H$, in which the elements of M induce the following automorphisms in $\bar{H}$:

$$m^{-1}h(r)m = h(r \cdot m\psi).$$

Obviously, $[\operatorname{Ker}\psi, \bar{H}] = E$ and

$$\Gamma/\operatorname{Ker}\psi \cong H \operatorname{wr} R.$$

Thus, finally, we obtain that $\mathfrak{A}_2$ contains any extension of any group $H \in \mathfrak{A}_1$ by means of any group $R \in \mathfrak{M}$, i.e. $\mathfrak{A}_1\mathfrak{M} \subseteq \mathfrak{A}_2$. But, on the other hand, as was shown above, an $\mathfrak{A}_2$-free group of countable rank can be represented as an extension of $V_1(\Phi) \in \mathfrak{A}_1$ by means of $G \in \mathfrak{M}$, and therefore $\mathfrak{A}_1\mathfrak{M} = \mathfrak{A}_2$, and the variety $\mathfrak{A}_2$ is decomposable, contrary to the supposition. Hence $\mathfrak{A}_1 = \mathfrak{A}_2$ and (by Lemma 3) $\mathfrak{B}_1 = \mathfrak{B}_2$. The assertion of the theorem is evident.

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Note: Figure translations are in progress. See original paper for figures.

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