Corresponding Member of the Academy of Sciences of the USSR A. V. POGORELOV

`![Fig. 1](#)`

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

THEORY OF ELASTICITY

Corresponding Member of the Academy of Sciences of the USSR A. V. POGORELOV

ON THE STABILITY OF AXISYMMETRIC DEFORMATIONS OF SPHERICAL SHELLS UNDER AXISYMMETRIC LOADINGS

Usually, when investigating supercritical deformations of shells of revolution under loads possessing axial symmetry, the deformations are also assumed to be axisymmetric. However, as experience shows, in reality this need not be the case. Thus, for example, if a spherical segment freely supported along its edge is loaded by a concentrated force applied at the center of the segment, then for a comparatively small value of this force the region of bulging has the form of a circular meniscus, while when the force reaches a certain critical value the boundary of the meniscus begins to take the form of a triangle with rounded vertices (Fig. 1). Another example: in testing spherical shells for stability under external pressure, after a "snap-through" the dent on the surface of the specimen has the form of a polygon rather than a circle, although both the shell and the mode of loading possessed perfect symmetry.

Fig. 1

In the present note some results will be given on the stability of axisymmetric deformations of spherical shells under two modes of loading: by a concentrated force and by uniform external pressure. General considerations concerning the method of investigation were set forth in the author' s previous publications, in particular in paper ([1]).

1. First of all we wish to note the very possibility of a loss of axial symmetry of the deformation. The point is that, as shown in ([1]), the supercritical deformation of a strictly convex shell fixed along its edge is, in a certain approximation, obtained as the mirror reflection of some segment of it. The proof of this assertion is based on two propositions: 1) elastic deformations of a shell are essentially geometrical bendings, 2) a strictly convex surface fixed along its edge does not admit bendings without loss of regularity. The application of this result to real shells is limited by two

Fig. 2

Figure 2: Fig. 2

conditions following from assumptions 1) and 2). Namely, the region of bulging must occupy a significant part of the whole surface of the shell, so that the geometrical condition of fixation of the edge is not weakened by deformation of the middle surface of the shell. The stresses in the material of the shell must not exceed the elastic limit, so that the relative deformations of the middle surface caused by them may be regarded as small. In both examples cited above, these conditions are not satisfied for the violations of axisymmetric deformations of a spherical shell. Therefore, the very deviation of the deformation from axial symmetry does not contradict the results obtained in ([1]).

2. We determine the state of elastic equilibrium of the shell by minimizing the functional

$$W(F) = U(F) - A(F),$$

which is considered on isometric transformations of the initial surface of the shell. $U(F)$ is the strain energy of the shell in the form F; $A(F)$ is the work done by the external load. The energy $U(F)$ over the main surface of the shell (excluding the edges) is determined in the usual way, and along the edges γ by the formula

$$U = \int_\gamma cE\delta^{5/2}\alpha^{5/2}k^{1/2}\,ds$$

(see (1)). Here 2α is the angle between the tangent planes to the surface F along the edge γ; k is the curvature of the edge; δ is the thickness of the shell; integration is along the arc of the edge γ. The constant $c \approx 0.18$. The principal difficulty in solving the problem of the elastic equilibrium of a shell under this method of consideration reduces to determining all isometric transformations of the middle surface of the shell.

Fig. 2

3. We restrict ourselves to considering such isometric transformations of the spherical surface in which an edge γ is formed along a curve given by the equation

$$r = R\rho\,(1 + \lambda \cos k\vartheta)$$

(r, ϑ are polar geodesic coordinates). A surface isometric to the sphere with such an edge will depend on two parameters ρ and λ. We find the form of this surface for sufficiently small values of ρ and λ. Further, under the assumption of such smallness of the parameters ρ and λ, we find $U(F)$ and $A(F)$. For the strain energy one obtains the expression

$$U(F) = 2\pi c E \delta^{5/2} R^{1/2} \rho^3 \left(1 + \lambda^2 \left(k^2 + \frac{(1+k^2)^2}{16}\right)\right) + \frac{\pi \rho^2 E \delta^3}{3} \lambda^2 (k^2 - 1) k.$$

The work A done by the external load under the action of a concentrated force f is

$$A(F) = f\rho^2 \left(1 + \frac{\lambda^2 k^2}{2}\right) R.$$

In the expressions $U(F)$ and $A(F)$ only the leading terms with respect to order of magnitude have been written out.

For a shell in a state of equilibrium, the parameters ρ and λ, which characterize the deformation, are determined from the system of equations

$$\frac{\partial}{\partial \rho}(U - A) = 0, \qquad \frac{\partial}{\partial \lambda}(U - A) = 0.$$

For fixed k, this system with respect to ρ and λ always has the solution $\lambda = 0$ (axisymmetric deformation). If ρ is sufficiently small, then this solution will be unique. This means that for small deformation the region of bulging has the form of a circle. Conversely, for large deformations (i.e., large ρ) the system admits a solution with $\lambda \neq 0$. The value

ρ, which separates these two cases, satisfies the system of equations

$$\left.\frac{\partial}{\partial \rho}(U - A)\right|_{\lambda=0} = 0, \qquad \left.\frac{1}{\lambda}\frac{\partial}{\partial \lambda}(U - A)\right|_{\lambda=0} = 0.$$

For $k = 3$ this gives

$$\rho = \frac{1}{c}\sqrt{\frac{\delta}{R}},$$

or, introducing the radius $r = R\rho$ of the circle of buckling,

$$r = \frac{1}{c}\sqrt{\delta R}.$$

Thus, **buckling in the form of a circle under the action of a concentrated force is stable as long as the radius of this circle is $r \leq \sqrt{\delta R}/c$. The critical force f is determined by the formula**

$$f = \frac{3\pi E\delta^3}{R}.$$

4. In Fig. 2 a graph is presented of the dependence of the critical force f on the thickness δ for copper shells of radius $R = 80$ mm. The points indicate the values of the critical force obtained experimentally.

5. Investigation of postcritical deformations of a spherical shell under uniform external pressure leads to the same conclusion as in Sec. 3 for a concentrated load. Namely, the postcritical deformation possesses axial symmetry, i.e., the buckling region has the form of a circle as long as the radius of the circle is $r \leq \sqrt{\delta R}/c$. After this the buckling region begins to assume the form of a triangle with rounded vertices.

Physical-Technical Institute of Low Temperatures
Academy of Sciences of the Ukrainian SSR

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1. A. V. Pogorelov, *On the theory of elastic shells under postcritical deformations*, Kharkov, 1960.

Note: Figure translations are in progress. See original paper for figures.