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Abstract

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UNIFORM DISTRIBUTION OF POINTS ON A SPHERE AND SOME ERGODIC PROPERTIES OF SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS IN THE COMPLEX DOMAIN

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Dense trajectories, ergodicity, and mixing occur frequently in analysis. The metric theory of dynamical systems (see, for example, ([1])) gives an approach to these questions, at least in the case of "one-parameter time." In the present note we consider some problems in which the role of time is played by a non-commutative discrete group. We were led to these problems by an attempt to study the ergodic properties of solutions of linear differential equations in the complex domain (see ([2])).

§ 1. **Uniform distribution of points on a sphere.**

Theorem 1. *Let A and B be two rotations of the sphere S^2, and let x be a point of the sphere. If the sequence of points*

$$x;\ Ax,\ Bx;\ A^2x,\ ABx,\ BAx,\ B^2x;\dots \tag{1}$$

is everywhere dense on the sphere S^2, then it is uniformly distributed.

By uniform distribution here the following is meant. Let Δ be a region on the sphere S^2 bounded by a piecewise smooth curve. By applying n rotations A, B to the point x, one can obtain 2^n points

$$A^nx,\ A^{n-1}Bx,\ A^{n-2}BAx,\dots,\ B^nx. \tag{2}$$

Denote by $p_n(\Delta)$ the number of points (2) that fall into the region Δ. Theorem 1 then asserts that

$$\lim_{n\to\infty}\frac{p_n(\Delta)}{2^n}=\frac{\operatorname{mes}\Delta}{\operatorname{mes}S^2}. \tag{3}$$

In proving Theorem 1 we use H. Weyl' s method ([3]). Consider an arbitrary continuous function $f(x)$ on the sphere S^2. Form the arithmetic mean $f_n(x)$ of the values of $f(x)$ at the points (2). Following Weyl, in order to prove (3) it suffices to establish that

$$\lim_{n\to\infty} f_n(x) = \bar{f} = \int_{S^2} f(x)\,dx \,/\, \operatorname{mes} S^2. \tag{4}$$

In the study of the time means f_n, unitary operators acting in $L_2(S^2)$ naturally arise:

$$Af(x) = f(A^{-1}x); \qquad Bf(x) = f(B^{-1}x).$$

With their help the time mean can be written in the form

$$f_n(x) = \frac{1}{2^n}(A+B)^n f(x) = \left(\frac{A+B}{2}\right)^n f(x). \tag{5}$$

As is known (see, for example, ([4])), the space $L_2(S^2)$ decomposes into an orthogonal sum of subspaces R_l $(l = 0, 1, 2, \ldots)$, invariant under all rotations of the sphere. The space R_l, of dimension $2l+1$, consists of spherical functions of degree l and has no invariant subspaces with respect to all rotations.

It is easy to see that it is enough to prove formula (4) for functions $f(x)$ belonging to each invariant subspace R_l.

Lemma 1. *Let A and B be finite-dimensional unitary operators. Then either for some $k \geq 1$*

$$\left\| \left(\frac{A+B}{2}\right)^k \right\| < 1, \tag{6}$$

or for some vector $f \neq 0$ we have

$$Af = Bf; \quad A^2 f = ABf = BAf = B^2 f; \quad A^3 f = A^2 Bf = ABAf = \ldots = B^3 f \tag{7}$$

and so on. The proof of the lemma is based on the fact that from $\|f+g\| = \|f\| + \|g\|$ and $\|f\| = \|g\| = 1$ it follows that $f = g$.

We now prove formula (4) for f from R_l, $l > 0$. If on R_l the operators A and B from (5) satisfy (6), then $f_n \to 0$ and (4) is proved. Let us show that case (7) is impossible. Indeed, from the density everywhere of the sequence (1) it easily follows that the closure of the products of A and B is the entire group of rotations of the sphere. Therefore (7) would imply the commutativity of the representation of this group in R_l, which does not occur for $l > 0$. Theorem 1 is proved.*

§ 2. Generalization. Theorem 1 may be regarded as an ergodic theorem in which the role of time is played by a free semigroup with two generators. One may construct dynamical systems in which the "time" is a group Γ with a finite number of generators $a_1, \dots, a_s$. We are concerned with the group of measure-preserving transformations A_γ $(\gamma \in \Gamma)$ of a space with measure Ω, for which

$$A_{\gamma_1\gamma_2} = A_{\gamma_1} A_{\gamma_2}$$

and

$$A_{\gamma^{-1}} = A_\gamma^{-1}.$$

To define time averages, consider the set Γ_n of elements of Γ obtained from $a_1, a_1^{-1}, \dots, a_s, a_s^{-1}$ by n (but not fewer than n) multiplications, and let their number be $N(n)$. Then the "time average" $f_n(x)$ of a function $f(x)$, $x \in \Omega$, is defined as

$$f_n(x) = \sum_{\gamma \in \Gamma_n} f(A_\gamma x)/N(n).$$

The method of § 1 makes it possible to investigate the behavior of $f_n(x)$ in certain cases to which the term "discrete spectrum" applies.

Let Ω be a homogeneous space (in § 1, the sphere S^2), on which a compact Lie group G acts transitively, and suppose the transformations A_γ $(\gamma \in \Gamma)$ belong to G. For a number of groups Γ one can prove that *the sequence of points $A_\gamma x$ is uniformly distributed in its closure if and only if it is connected*; in other words, the time averages $f_n(x)$ of a continuous function converge to the space average over the closure of the trajectory $A_\gamma x$ $(\gamma \in \Gamma)$.

As an example we consider 2 cases:

1) Γ is a free group with two generators a, b.

2) Γ is the group with generators a, b, c and relation $abc = e$.

It is easy to see that $f_n(x) = S_n f(x)$, where $S_0 = E$ and, respectively,

$$S_1 = \frac{1}{4}(A + B + A^{-1} + B^{-1}); \quad S_{n+1} = \frac{4}{3} S_1 S_n - \frac{1}{3} S_{n-1} \quad (n \geq 1), \tag{8}$$

$$S_1 = \frac{1}{6}(A + B + C + A^{-1} + B^{-1} + C^{-1}); \quad S_{n+1} = \frac{3}{2} S_1 S_n - \frac{1}{4} S_n - \frac{1}{4} S_{n-1} \quad (n \geq 1) \tag{9}$$

(here A, B, C are unitary operators corresponding to the generators of the group Γ: $Af(x) = f(A_a^{-1} x)$).

Consider the closure of the trajectory $A_\gamma x$ $(\gamma \in \Gamma)$. This is a homogeneous space M, on which the closure $\overline{\Gamma}$ of the group A_γ in G acts transitively. Decompose $L_2(M)$ into the orthogonal sum of finite-dimensional subspaces R_l, invariant and irreducible with respect to $\overline{\Gamma}$ (see [4]).

The operators S_1 (and consequently S_n as functions of them) are self-adjoint. The study of S_n in R_l reduces to the study of their eigenvalues, for which from (8) or (9) recurrence equations are obtained. Solving these equations, we see that either in R_l, $S_n \to 0$ $(n \to \infty)$, or R_l is one-dimensional and for $f \in R_l,\ D \in \overline{\Gamma}$ we have $Df = \pm f$.

We now show that f is constant on M, if M is connected. Denote by K the component of the identity in $\overline{\Gamma}$. For D in K, obviously, $Df = f$, hence on Kx the function f is constant. But since M is connected, it coincides with Kx, and on it $f = \text{const}$. From this it follows easily that the time averages tend to the space average.

* Another proof of Theorem 1 was given by M. Malyutov.

Remark 1. Above we studied time averages over the "spheres" Γ_n. It is easy to derive analogous theorems on averages over the "balls" $\bigcup_{k=0}^{n} \Gamma_k$.

Remark 2. If A and B are two rotations of the sphere taken at random, then in the general case the sequence (1) is everywhere dense. It is probable that, for two elements A and B of a compact Lie group G, the general case will be everywhere dense in G for products of A and B. If, however, the group G is not compact, then the subgroup generated by any number of elements may be properly discrete (example: G is the group of motions of the Lobachevsky plane, Γ is a subgroup of a discrete group associated with a surface of genus p (see (5))).

§ 3. **Equations with complex time.** From the geometric point of view, solutions of ordinary differential equations in the complex domain are represented by two-dimensional surfaces on which the phase space is fibered. Such a surface may wind everywhere densely around the phase space or around a part of it. In this case it is natural to expect, in some sense, a uniform distribution.

Consider the system of linear differential equations

$$\frac{dx}{dz} = A(z)x, \tag{10}$$

where z is a complex variable, x is a vector $(x_1, \dots, x_n)$ of the n-dimensional complex space C_n, and A is a matrix depending analytically on z, except for three special points z_1, z_2, z_3 on the Riemann sphere.

The phase space of real dimension $2n + 2$ is the direct product of the Riemann sphere without the three points z_1, z_2, z_3, denoted below by Z, and $C_n(z)$. It is fibered into solutions–surfaces of real dimension 2, locally given by the equation $x = x(z)$, where $x(z)$ satisfies the system (10), and $z \in Z$.

To each path on Z issuing from z_0, and to each vector $x_0 \in C_n(z_0)$, there corresponds a unique solution $x(z)$ with the initial condition $x_0(z_0)$. Thus a linear mapping of $C_n(z_0)$ onto $C_n(z)$ is defined. In particular, to a closed path γ there corresponds a linear transformation A_γ of the space $C_n(z_0)$ into itself. The transformation A_γ depends only on the homotopy class of the path γ in Z; these transformations form an antirepresentation of the fundamental group of Z. The group of transformations A_γ is called the monodromy group of the system (10).

Lemma 2. *If the monodromy group is bounded, then the system (10) has a single-valued first integral* $(B(z)x, \bar{x}) = \text{const}$*, where* $B(z)$ *is a positive definite self-adjoint matrix, single-valued for* $z \in Z$*.*

The proof is based on the fact that, in view of the compactness of the closure of A_γ, the representation A_γ is equivalent to a unitary one.

It follows from Lemma 2 that every two-dimensional surface representing a solution in the $(2n+2)$-dimensional space remains constantly on the $(2n+1)$-dimensional surface $(Bx, \bar{x}) = c$, and the points of different branches of the solution $x(z)$ over one point z_0 lie on the sphere $(B(z_0)x, \bar{x}) = c$. According to the results of §§ 1, 2, these points are uniformly distributed in their closure (if it is connected): the fundamental group of Z has three generators a, b, c and the relation $abc = e$.

There is a case when one can write explicitly the condition for boundedness of the monodromy group. This is Gauss' s hypergeometric equation

$$z(1-z)\frac{d^2x}{dz^2} + \left[\gamma - (\alpha+\beta+1)z\right]\frac{dx}{dz} - \alpha\beta x = 0. \tag{11}$$

We shall assume that the parameters α, β, γ are real.

Theorem 2. *The hypergeometric equation* (11) *has a single-valued first integral*

$$b_{11}x\bar{x} + b_{12}x\bar{x}' + b_{21}x'\bar{x} + b_{22}x'\bar{x}' = \text{const}, \tag{12}$$

where $x' = dx/dz$*, and* $b_{ij}(z)$ *are single-valued (but not complex-analytic) functions defined for* $z \neq 0, 1, \infty$ *and forming a self-adjoint matrix* $\|b_{ij}(z)\|$*.*

To find the functions b_{ij}, it is enough to observe that both generators A_a, A_b of the monodromy group, written explicitly in (7), preserve a certain scalar product.

According to Riemann and Schwarz, with equation (11) there is associated a triangle bounded by arcs of circles, with angles $\lambda\pi, \mu\pi, \nu\pi$, where $\lambda = |1-\gamma|$, $\mu = |\gamma - \alpha - \beta|$, $\nu = |\alpha - \beta|$ (see (6)). *If the sum of the angles of this triangle is greater than* π*, then the matrix* $\|b_{ij}(z)\|$ *is positive definite, the monodromy group consists of matrices unitary in the metric* $\|x\| = (B(z_0)x, \bar{x})$*, and all branches of the solution over each point* z *lie on the sphere* (12) *in the space* x, x' *and are uniformly distributed on this sphere for almost all values of the parameters* α, β, γ (the exceptional values constitute a one-dimensional manifold).

§ 4. In conclusion we indicate several unsolved questions.

1°. Are the ergodic theorems of Birkhoff and Neumann valid for dynamical systems with noncommutative time of the type of § 2?

2°. Do the results of § 2 extend to arbitrary groups Γ with a finite number of generators?

3°. Do the results of §§ 1, 2 extend to the noncompact case? (For example, let Ω be the Euclidean plane or the Lobachevsky plane.)

4°. What is the generalization of §§ 1, 2 to the case when the role of time is played by a Lie group, for example the group of motions of the Lobachevsky plane?

5°. Equations (10) can be written in the form $dx = (A(z)\,dz)x$. If by $A(z)\,dz$ one understands a matrix of differentials, then the arguments of § 3 carry over to equations on Riemann surfaces. The difficulty consists in determining whether the monodromy group is bounded.

6°. The uniform distribution of the surface representing the solutions (10) in the $(2n+1)$-dimensional manifold $M_c : (Bx, \bar{x}) = c$ probably takes place with respect to the following metric: on Z one introduces a metric of constant negative curvature (see (5)), and on $C_n(z)$ the metric is determined by the scalar product $(B(z)x, y)$.

7°. The system (10) can be regarded as a dynamical system in which the role of time is played by the universal covering Z, i.e. the Lobachevsky plane. But with it one can also associate an ordinary dynamical system with continuous time. To this end, take as a point of the new phase space the point $(z, x) \in M_c$ together with the direction ξ of a vector tangent to Z at z. Define the motion as follows: the point z moves uniformly along the geodesic direction ξ, while x over z moves in accordance with equations (10). The metric and invariant measure are defined in 6°.

The indicated construction makes it possible to "multiply" a flow given on a manifold by a group of automorphisms (which is a representation of the fundamental group of the manifold). It is of interest to study the resulting "products."

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Note: Figure translations are in progress. See original paper for figures.