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V. A. DUBROVSKII

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Abstract

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V. A. DUBROVSKII

QUASICLASSICAL ASYMPTOTICS OF THE DIRAC EQUATION

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F. Klein expressed the idea of replacing problems connected with the classical Hamilton–Jacobi equation by problems connected with a wave equation for which the Hamilton–Jacobi equation is the equation of characteristic manifolds ([1]). This idea was developed by Rumer as applied to a number of equations of wave mechanics ([2]). In the present work this idea is used to construct the quasiclassical asymptotics of the most interesting equation of quantum mechanics —the Dirac equation.

The solution of the relativistic Hamilton–Jacobi equation

$$-\left(\frac{\partial s}{\partial t}+e\Phi\right)^2+c^2\left(\operatorname{grad} s-\frac{e}{c}A\right)^2+m^2c^4=0 \tag{1}$$

can be sought in implicit form: $\varphi(x,y,z,s,t)=0$. Then the equation takes the form

$$-\left(\frac{\partial \varphi}{\partial t}-e\Phi\frac{\partial \varphi}{\partial s}\right)^2+c^2\left(\operatorname{grad}\varphi+\frac{e}{c}A\frac{\partial \varphi}{\partial s}\right)^2+m^2c^4\left(\frac{\partial \varphi}{\partial s}\right)^2=0, \tag{2}$$

where s is the action, and $\{\Phi, A\}$ is the 4-vector of the potential. If we consider a family of solutions $\varphi=C$, depending on the parameter C and including the solution $\varphi=0$, then equation (2) will be a partial differential equation in the variables x,y,z,s,t. To equation (2) one can associate a wave equation for which (2) is the equation of characteristic manifolds:

$$\mathcal{L}[u]=\left(\frac{\partial}{\partial t}-e\Phi\frac{\partial}{\partial s}\right)u-\alpha^\nu\left(c\frac{\partial}{\partial x_\nu}+eA_\nu\frac{\partial}{\partial s}\right)u+mc^2\beta u_s=0, \tag{3}$$

where α^ν, β are the known four-row matrices used in writing the Dirac equation ([3]), and $u=\{u^1,u^2,u^3,u^4\}$. The characteristic determinant of this system is equal to the square of the left-hand side of equation (2).

We note that the equation

$$\mathcal{L}[u] = -\left(\frac{\partial}{\partial t} - e\Phi\frac{\partial}{\partial s}\right)^2 u + c^2\left(\frac{\partial}{\partial x_\nu} + \frac{e}{c}A_\nu\frac{\partial}{\partial s}\right)^2 u + m^2c^4\frac{\partial^2 u}{\partial s^2} = 0 \qquad (3')$$

has equation (2) as its equation of characteristic manifolds.

We shall consider equation (3). For the boundary condition with respect to the fifth coordinate s, we adopt the periodicity condition with period h (Planck' s constant). The following considerations may be adduced in justification of this. The result of measuring a segment Δx by means of a light signal is expressed in the form $\Delta x - c\Delta t = 0$, where Δt is the signal transit time and c is the velocity of propagation of the signal. Hence, under the condition of invariance of this method of measurement in all inertial reference frames, the known relations of relativistic dynamics are obtained. However, in view of the wave character of the signal, one must take into account that the length of the segment is determined with an accuracy up to the wavelength, i.e. $\Delta x - c\Delta t \simeq \pm\lambda$. Taking into account the relations $E = h\nu$ and $E = pc$, we obtain $p\Delta x - E\Delta t \simeq \pm h$, i.e., even using the most perfect signal, which is light, we never can

same action, smaller than h. And, as in electromagnetic phenomena there is periodicity in time, so in the present case one should postulate periodicity with respect to the fifth coordinate s with period h.

Now, proceeding from the method of separation of variables, one should seek the solution of equation (3) in the form

$$u = \psi(x, y, z, t)e^{-is/\hbar}, \qquad (4)$$

where $\hbar = h/2\pi$, and ψ satisfies the Dirac equation

$$\left[i\hbar\frac{\partial}{\partial t} - e\Phi - i\hbar c\alpha^\nu\frac{\partial}{\partial x_\nu} - eA_\nu\alpha^\nu + mc^2\beta\right]\psi = 0. \qquad (5)$$

Thus, mathematically, the relation between quantum mechanics and classical dynamics is expressed in the connection of equation (3) with its characteristic equation (2). Hence follow the well-known relations

$$E = \hbar\omega, \qquad \mathbf{p} = \hbar\mathbf{k}.$$

Indeed, suppose the potentials Φ, A_ν do not depend on time. Then there appears a conserved quantity, called energy, and the equation for φ will be

$$(E - e\Phi)^2(\varphi_s)^2 + c^2\left(\operatorname{grad}\varphi + \frac{e}{c}\mathbf{A}\varphi_s\right)^2 + m^2c^4(\varphi_s)^2 = 0. \qquad (6)$$

Correspondingly, instead of (5) we obtain

$$\left[E - e\Phi - i\hbar c\alpha^{\nu}\frac{\partial}{\partial x_{\nu}} - eA_{\nu}\alpha^{\nu} + mc^2\beta\right]\psi = 0.$$

If, however, the separation of variables is carried out in equation (5) (the corresponding conserved quantity is called frequency), then we obtain

$$\left[\hbar\omega - e\Phi - i\hbar c\alpha^{\nu}\frac{\partial}{\partial x_{\nu}} - eA_{\nu}\alpha^{\nu} + mc^2\beta\right]\psi = 0.$$

Since both these equations describe one and the same process, we must equate E and $\hbar\omega$. Analogous arguments are valid with respect to the second of relations (6). It is not difficult to verify that the same is also true for a nonrelativistic particle and for light, i.e., the use of the relation $E = h\nu$ is legitimate and is not a departure beyond the framework of this method. A partly similar derivation of equations (6) is given for the particular case of the Schrödinger equation with a time-independent potential in ([4]).

We shall obtain an asymptotic solution of the Dirac equation from the solution of equation (3), taking (4) into account. Thus, we shall seek the solution of equation (3) with initial condition

$$u(x, y, z, s, 0) = u_0(x, y, z, s). \tag{7}$$

For this we use the method that was developed as applied to the equations of acoustics ([5]), Maxwell' s equations ([6,7]), and in the theory of elasticity ([8–10]). Following the idea of the method, we shall seek the solution in the form

$$u(x, y, z, s, t) = \sum_{k=0}^{\infty} w_k(\varphi(x, y, z, s, t))\, g^k(x, y, z, s, t), \tag{8}$$

where $w_k(\varphi)$ is a system of generalized functions ([11]) such that $w_k'(\varphi) = w_{k-1}(\varphi)$ (see ([12])). Correspondingly, we represent the initial condition (7) in the form

$$u_0(x, y, z, s) = \sum_{k=0}^{\infty} w_k(\varphi_0(x, y, z, s))\, g_0^k(x, y, z, s); \tag{9}$$

here $\varphi_0(x, y, z, s) = c$ is the initial manifold on which the initial data are prescribed. Substituting (8) into (3) and equating the coefficients of w_k to zero, we obtain the system of equations

$$Bg^0 \equiv \left[I\varphi_t - c\alpha^{\nu}\varphi_{x_{\nu}} + \left(-Ie\Phi - \alpha^{\nu}eA_{x_{\nu}} + \beta mc^2\right)\varphi_s\right] g^0 = 0; \tag{10}$$

$$Bg^{k+1} + \mathcal{L}\,[g^k] = 0, \tag{11}$$

where I is the identity matrix. Consequently,

$$\det|B| = \Big[-(\varphi_t - e\Phi\varphi_s)^2 + (c\varphi_{x_\nu} + eA_\nu\varphi_s)^2 + m^2c^4\varphi_s^2\Big]^2 = 0. \tag{12}$$

The fourth-order matrix B has rank equal to two. Let l^α and r^α, $\alpha = 1,2,3,4$, be vectors such that $l^\alpha B = 0$, $Br^\alpha = 0$. Then from (10) it follows that

$$g^0 = \sum_\alpha \sigma^{0\alpha} r^\alpha \quad (\alpha = 1,2).$$

A necessary and sufficient condition for compatibility of system (11) will be $l^\alpha \mathcal{L}\,[g^k] = 0$, and hence, for $v = 0$, an equation for $\sigma^{0\alpha}$ is obtained:

$$l^\alpha B^i r^\alpha \frac{\partial}{\partial x_i}\sigma^{0\alpha} + l^\alpha \mathcal{L}\,[r^\alpha]\sigma^{0\alpha} = 0, \tag{13}$$

where $B^t = I$, $B^\nu = -c\alpha^\nu$, $B^s = -Ie\Phi - \alpha^\nu eA_\nu + \beta mc^2$, $x_i = x, y, z, s, t$. In this equation $\sigma^{0\alpha}$ are differentiated in the bicharacteristic direction, i.e., along the trajectory (see (12)).

If (12) is written in the form

$$\{\varphi_t - e\Phi - [m^2c^4\varphi_s^2 + (c\varphi_{x_\nu} - eA_\nu\varphi_s)]^{1/2}\}^2\{\varphi_t - e\Phi +$$

$$+[m^2c^4\varphi_s^2 + (c\varphi_{x_\nu} - eA_\nu\varphi_s)]^{1/2}\}^2 = 0, \tag{14}$$

then it is immediately clear that it determines two double roots $\varphi_t^\pm$, i.e. through the initial manifold $\varphi_0(x,y,z,s) = c$ there pass four pairwise glued characteristic surfaces. The direction of differentiation in (13) is determined by the characteristics that form these surfaces, the parameter of differentiation along the trajectory being the time. Physically, the characteristics lying on the two glued characteristic surfaces correspond to the trajectories of a particle and an antiparticle, while the two vectors belonging to each of the roots $\varphi_t^\pm$ distinguish the two spin directions.

The solution of equation (13) is

$$\sigma^{0\alpha} = \left[\frac{\partial(x^0y^0z^0)}{\partial(xyz)}\right]^{1/2}\sigma_0^{0\alpha}, \tag{15}$$

where $\sum_{\alpha=1}^{2}|g_0^{0\alpha}|$ is constant along the trajectory and is determined through the initial data from the equation

$$\sum_{\alpha}\sigma^{0\alpha}(x,y,z,0)\,r^{\alpha}(x,y,z,0)=g_0^0(x,y,z). \tag{16}$$

We shall seek the solution of system (11) with respect to g^{k+1} in the form

$$g^{k+1}=\sum_{\alpha}\sigma^{k+1\alpha}r^{\alpha}+h^{k+1}, \tag{17}$$

where h^{k+1} is any particular solution of the system. It is known if g^k is known. $\sigma^{k\alpha}$ is determined from the equation

$$l^{\alpha}B^{i}r^{\alpha}\frac{\partial\sigma^{k\alpha}}{\partial x_i}+l^{\alpha}\mathcal{L}\,[r^{\alpha}]\sigma^{k\alpha}+l^{\alpha}\mathcal{L}\,[h^{k\alpha}]=0 \tag{18}$$

and is known if $h^{k\alpha}$ is known. The unique solution is ensured by the initial data with the aid of equations analogous to (16):

$$\sum_{\alpha}\sigma^{k\alpha}r^{\alpha}=g_0^k-\sum_{\alpha}h^{k\alpha}.$$

Thus, one can successively find $\sigma^{0\alpha}$, $h^{1\alpha}$ from (11) for $k=0$, $\sigma^{1\alpha}$ from (18) for $k=1$, and so on; moreover, $h^{k\alpha}$ is always known if $\sigma^{k-1\,\alpha},\sigma^{k-2\,\alpha},\dots,\sigma^{0\alpha}$ are known.

Since the transition from equation (3) to the Dirac equation (5) is carried out by means of (4), one should choose, as $w_k(\varphi)$,

$$\left(\frac{\hbar}{i}\right)^{k}\exp\left\{\frac{i}{\hbar}\varphi(x,y,z,s,t)\right\}.$$

The solution of equation (2) $\varphi(x,y,z,s,t)=c$ can be written in the form $-s+s(x,y,z,t)=c_1$ (the potentials Φ, A_ν are assumed not to depend on s).

Let the initial conditions be given as follows:

$$u_0(x,y,z)=g_0(x,y,z)\exp\left\{\frac{i}{\hbar}\varphi\right\},$$

i.e., all g_0^k for $k\geqslant 1$ are equal to zero. Then the solution of equation (3) takes the form

$$u=e^{-\frac{i}{\hbar}s}\,\psi(x,y,z,t),$$

where $\psi(x,y,z,t)$ is an asymptotic solution of the Dirac equation, and $\psi = \psi^+ + \psi^-$,

$$\begin{aligned} \psi^+ &= \sqrt{D}\,\exp\left\{\frac{i}{\hbar}s(x,y,z,t)\right\}\left[\sum_{\varkappa=1,2} g_0^{+\varkappa} r^{+\varkappa} + \sum_{k=1}^{\infty}\left(\frac{\hbar}{i}\right)^k g^{k+}\right], \\ \psi^- &= \sqrt{D}\,\exp\left\{-\frac{i}{\hbar}s(x,y,z,t)\right\}\left[\sum_{\varkappa=1,2} g_0^{-\varkappa} r^{-\varkappa} + \sum_{k=1}^{\infty}\left(\frac{\hbar}{i}\right)^k g^{k-}\right]; \end{aligned} \tag{19}$$

here $D = \partial(x^0y^0z^0)/\partial(xyz)$, and g^k are determined by means of the recurrence formulas (18) and (17). The plus and minus indices correspond to two glued characteristic surfaces (particle and antiparticle), and $\varkappa = 1,2$ distinguishes the two multiple roots (spin direction). Accordingly, one should sum over α in formula (17).

The asymptotic solution of the equation obtained as a result of substituting (4) into (3′) has a form completely analogous to the solution (19), except that the spin variables are absent.

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Note: Figure translations are in progress. See original paper for figures.