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Abstract

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MATHEMATICAL PHYSICS

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ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF SOME NONLINEAR PROBLEMS IN THE THEORY OF NUCLEAR REACTORS

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Some problems in the theory of nuclear reactors lead to the solution of a nonlinear transport equation, which for simplicity we shall write, in the monoenergetic case with isotropic scattering, in the form

$$\lambda\varphi(P) = \frac{1}{4\pi}\int_V \frac{dP'}{|P-P'|^2}\exp\left\{-\int_0^{|P-P'|}\alpha\left[P' + \frac{P-P'}{|P-P'|}s,\ B\varphi\right]ds\right\}\times$$

$$\times\beta\left[P',\ B\varphi(P')\right]\varphi(P'). \tag{1}$$

Here λ is a parameter; $\varphi(P)$ is the neutron flux; α and β are functions depending on $B\varphi(P)$, where B is a certain operator, and characterizing the absorbing and multiplying properties of the reactor materials; the integration is carried out over the three-dimensional region V occupied by the reactor.

For illustration, let us restrict ourselves to two examples. Suppose it is required to construct a fast-neutron reactor with a prescribed law of spatial variation of heat release per unit volume of coolant. Let $\rho(P)$ be the volume fraction of fuel (fissile substance); $1-\rho(P)$ the volume fraction of coolant, and $Q(P)$ the prescribed law of heat release, referred to a unit volume of coolant. The requirements on the reactor can be expressed in the form

$$\frac{\sigma_f(P)\rho(P)\varphi(P)}{1-\rho(P)} = Q(P), \tag{2}$$

where $\sigma_f(P)$ is the fission cross section. Introduce the notation

$$Q/\sigma_f = F(P) \geq 0.$$

Then

$$\rho(P) = \frac{F(P)}{\varphi(P) + F(P)}.$$

The total cross section $\alpha(P)$ can be written in the form

$$\alpha(P) = \alpha_0(P)(1-\rho) + \alpha_1(P)\rho = \frac{\alpha_0\varphi + \alpha_1 F}{\varphi + F}. \quad (3)$$

Similarly,

$$\beta(P) = \frac{\beta_0\varphi + \beta_1 F}{\varphi + F}. \quad (4)$$

Thus, in this example the operator B is defined by expressions (3), (4).

As a second example, consider a nonlinear relation between the temperature of the materials of a power reactor and the neutron flux. Heat release at a given point of the reactor is determined by the number of fissions at this point, which depends on the neutron flux. The neutron flux, in turn, depends on the cross sections for the interaction of neutrons with the reactor materials, which, owing to temperature effects, themselves depend on the flux. Thus in any reactor there is a nonlinear relation between temperature and neutron flux. In this case the operator B is an operator by means of which

from the flux one obtains the temperature T:

$$T(P) = B\varphi. \quad (5)$$

This operator can take into account heat transfer by heat conduction, convection, etc.

One can arrive at an equation of type (1) also in solving other problems of nuclear-reactor theory, as well as problems on the passage of high-power radiation through matter (in the latter case the equation will contain a term describing the radiation source).

Let us write (1) in the form

$$A\varphi = \lambda\varphi. \quad (6)$$

We note some properties of the nonlinear operator A. The operator A is continuous from the space of square-integrable functions $\mathcal{L}_2$ into $\mathcal{L}_2$. This property can be proved under the following assumptions: $\alpha(P, B\varphi)$ and $\beta(P, B\varphi)$ are almost everywhere positive and bounded as functions of two arguments and continuous as functions of the second argument, and the operator B is continuous. We note that these restrictions are physically justified.

Let us obtain conditions sufficient for the operator A to be u_0-concave.* Since for any $\varphi \in K$, $A\varphi > 0$, any positive constant may be taken as u_0.

Let us note first that instead of (6) one may consider the equation

$$\lambda^k \varphi = A^k \varphi. \tag{7}$$

A solution of (6) satisfies (7). If, however, the solution of (7) is unique, then it is the unique solution of (6). In addition, note that if an eigenvector of (6) exists, then the ratio of its maximum to its minimum must not exceed some constant r. This follows, for example, from the boundedness of the kernel of the operator A.** The latter circumstance allows us to seek the solution of (1) in the narrower cone K_r, which is formed by the functions $\varphi \in K$ satisfying the condition

$$\operatorname{vrai}\max_P \varphi(P) \le r \operatorname{vrai}\min_P \varphi(P).$$

In what follows we shall assume that A is a monotone operator. For this it is sufficient that α decrease monotonically, while $\beta\varphi$ increase monotonically as φ grows.

We now obtain a condition for the u_0-concavity of the operator A (if A is u_0-concave and monotone, then A^2, A^3, etc., possess these properties).

Theorem 1. *The operator A on the cone K_r is u_0-concave, at least under the condition*

$$\frac{\delta\beta}{\delta\alpha}(P) > r \operatorname{vrai}\sup_{P,\varphi} \frac{\beta}{\alpha}$$

for all $t \in (0, 1)$ and all $\varphi \in K_r$, where $\delta\beta > 0$ and $\delta\alpha > 0$ are the increments of β and α when φ is replaced by $t\varphi$.

For the proof it is enough to note that the operator A will be u_0-concave if

$$\int v \, d\Omega > 0,$$

where v satisfies the equation

$$\vec{\Omega}\nabla v + \alpha(P, Bt\varphi)v$$

$$= \delta\beta \cdot \varphi - \delta\alpha \cdot \int d\xi \, \exp\left\{ -\int_0^\xi \alpha[P - \xi'\vec{\Omega}, B\varphi] \, d\xi' \right\} \beta(P - \xi\vec{\Omega}, B\varphi)\varphi,$$

where

$$\vec{\Omega} = \frac{P - P'}{|P - P'|};$$

the integration is performed over the entire domain.

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* A is called u_0-concave (1) if there exists such a nonzero element $u_0 \in K$, where K is the cone of nonnegative functions, that for any nonzero $x \in K$ the inequalities

$$\alpha u_0 \leq Ax \leq \beta u_0, \qquad \alpha, \beta > 0,$$

hold, and if for every such $x \in K$ that

$$\alpha_1(x) u_0 \leq x \leq \beta_1(x) u_0, \qquad \alpha, \beta > 0,$$

it is true that

$$A(t_0 x) \geq [1 + \eta(x,t)]\, t_0 A x, \qquad t_0 \in (0,1), \quad \eta > 0.$$

** Here, by the kernel of the operator A is meant the expression

$$\frac{\exp\left\{-\int_0^{|P-P'|} \alpha\left[P' + s\,\frac{P-P'}{|P-P'|}, B\varphi\right] ds\right\}}{|P-P'|^2}\,\beta.$$

To fulfill this condition it is sufficient to require that

$$\delta\beta \cdot \varphi > \delta\alpha \cdot \int d\xi \, \exp\left[-\int_0^{\xi} \alpha \, d\xi'\right] \beta\varphi,$$

from which the assertion of the theorem follows.

On the basis of the results of work (1), one can formulate the following theorem.

Theorem 2. *Suppose the conditions ensuring the continuity, monotonicity, and u_0-concavity of the operator A are satisfied. Then the eigenvectors $\varphi(\lambda)$ of the operator A form a continuous branch of infinite length, and its positive spectrum forms the interval $(\lambda_\infty, \lambda_0)$, with*

$$\lim_{\lambda \to \lambda_0} \|\varphi(\lambda)\| = 0_\theta, \qquad \lim_{\lambda \to \lambda_\infty} \|\varphi(\lambda)\| = \infty.$$

The vector-function $\varphi(\lambda)$ is then unique and decreases as λ increases. The successive approximations $\varphi_n = \frac{1}{\lambda} A \varphi_{n-1}$ converge to the unique nonzero solution of equation (6) for any nonzero initial approximation $\varphi_0 \in K$. Here λ_0 is the eigenvalue of the strong Fréchet derivative of the operator A at zero, and λ_∞ is the eigenvalue of the strong asymptotic derivative corresponding to positive eigenfunctions.

The solution of equation (7) has analogous properties.

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REFERENCES

1. M. A. Krasnosel' skii, *Positive Solutions of Operator Equations. Chapters of Nonlinear Analysis*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.