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Abstract

Full Text

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ON THE ERROR OF NUMERICAL INTEGRATION

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In this paper an unimprovable, in the sense of order, estimate is obtained for the error of numerical integration for a certain class of functions. The methods for obtaining the results are the same as in the works of N. M. Korobov ([1,2]).

The functions $f(x_1, \dots, x_s)$ are assumed to be periodic, with period equal to one in each of the variables, and to be expandable in an absolutely convergent Fourier series:

$$f(x_1, \dots, x_s) = \sum_{m_1, \dots, m_s = -\infty}^{\infty} c(m_1, \dots, m_s) \exp\left[2\pi i (m_1 x_1 + \dots + m_{sx} s)\right]. \qquad (1)$$

If the Fourier coefficients satisfy the inequality

$$|c(m_1, \dots, m_s)| \leq (\overline{m}_1 \cdots \overline{m}_s)^{-\alpha} \qquad (\alpha > 1,\ \overline{m} = \max(1, |m|))$$

or the inequality

$$|c(m_1, \dots, m_s)| \leq \overline{m}_1^{-\alpha} (\overline{m}_2 \cdots \overline{m}_s)^{-\alpha_1}, \qquad (\alpha, \alpha_1 > 1),$$

then we shall say that the function belongs, respectively, to the class E_s^α or to the class E_s^α.

Lemma 1. *The number of solutions of the inequality*

$$\overline{m}_1 \cdots \overline{m}_s [\ln(1 + \overline{m}_2) \cdots \ln(1 + \overline{m}_s)]^2 \leq q \qquad (2)$$

in integers $m_1, \dots, m_s$ does not exceed $c_1(s) q$.

Proof by induction.

Lemma 2. *Let $p > s$ be a prime. There exists an integer a $(1 < a < p)$ such that the congruence*

$$m_1 + a m_2 + \dots + a^{s-1} m_s \equiv 0 \pmod{p}$$

has no solutions, except the trivial one, in the domain (2) with $q \le [sc_1(s)]^{-1}p$.

Proof. Introduce the notation

$$\delta_p(n) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod p, \\ 0, & \text{if } n \not\equiv 0 \pmod p. \end{cases} \quad (3)$$

The lemma asserts that for some a

$$\sum_{(2)}{}' \delta_p(m_1 + am_2 + \dots + a^{s-1}m_s) = 0,$$

where $\sum'_{(2)}$ denotes summation over the domain (2), excluding the set $(m_1, \dots, m_s) = (0, \dots, 0)$. We choose a from the condition

$$\sum_{(2)}{}' \delta_p(m_1 + am_2 + \dots + a^{s-1}m_s) = \min_{1 \le z < p} \sum_{(2)}{}' \delta_p(m_1 + zm_2 + \dots + z^{s-1}m_s). \quad (4)$$

It is easy to show that the right-hand side of (4) is equal to 0. Indeed,

$$\min_{1 \le z \le p} \sum_{(2)}{}' \delta_p(m_1 + zm_2 + \dots + z^{s-1}m_s) \le \frac{1}{p} \sum_{z=1}^{p} \sum_{(2)}{}' \delta_p(m_1 + \dots + z^{s-1}m_s) \le$$

$$\le \frac{s-1}{p} \sum_{(2)}{}' 1 < 1.$$

Corollary. Obviously, the assertion of Lemma 2 remains valid if the domain (2) is replaced by the domain

$$\overline{m}_1(\overline{m}_2 \dots \overline{m}_s)^{1+\varepsilon} \le q \le c_2(\varepsilon, s)p, \qquad (\varepsilon > 0).$$

Lemma 3. For $\alpha > 1,\ \varepsilon > 0$, the estimate

$$\sum_{\overline{m}_1(\overline{m}_2 \dots \overline{m}_s)^{1+\varepsilon} > q} \overline{m}_1^{-\alpha}(\overline{m}_2 \dots \overline{m}_s)^{-\alpha(1+\varepsilon)} \le c_3(\alpha, s, \varepsilon)q^{-(\alpha-1)}. \quad (5)$$

is valid.

Proof by induction.

Theorem 1. Let a be chosen as in Lemma 2. For functions $f \in \overline{E}_s^\alpha$ the quadrature formula

$$\int_0^1 \cdots \int_0^1 f(x_1, \dots, x_s)\, dx_1 \cdots dx_s - \frac{1}{p} \sum_{k=1}^{p} f\left(\frac{1}{p}k, \frac{a}{p}k, \dots, \frac{a^{s-1}}{p}k\right) = R,$$

where $|R| \le c(\alpha, \alpha_1, s) p^{-\alpha}$, is valid.

Proof. According to (1) and (3), the error of numerical integration can be written in the following form:

$$|R| = \left| \frac{1}{p} \sum_{m_1, \dots, m_s = -\infty}^{\infty} {}' c(m_1, \dots, m_s) \sum_{k=1}^{p} \exp\left[2\pi i \, \frac{m_1 + \cdots + a^{s-1} m_s}{p} \, k\right] \right| \le$$

$$\le \sum_{m_1, \dots, m_s = -\infty}^{\infty} {}' \overline{m}_1^{-\alpha} (\overline{m}_2 \dots \overline{m}_s)^{-\alpha\alpha_1} \delta_p(m_1 + a m_2 + \cdots + a^{s-1} m_s), \qquad (6)$$

where $\sum'$ denotes summation over the sets $(m_1, \dots, m_s) \ne (0, \dots, 0)$.

Applying Abel' s transformation successively with respect to each of the summation variables, we obtain

$$|R| < (\alpha\alpha_1)^s \sum_{m_1, \dots, m_s = 1}^{\infty} m_1^{-\alpha-1} (m_2 \dots m_s)^{-\alpha\alpha_1 - 1} \times$$

$$\times \sum_{|k_1| \le m_1, \dots, |k_s| \le m_s} {}' \delta_p(k_1 + a k_2 + \cdots + a^{s-1} k_s). \qquad (7)$$

Let us estimate from above the quantity

$$\sum_{|k_1| \le m_1, \dots, |k_s| \le m_s} {}' \delta_p(k_1 + a k_2 + \cdots + a^{s-1} k_s).$$

Obviously, it can be represented as a sum of 2^s terms, in each of which every summation variable has constant sign. All terms are estimated in the same way. Therefore we consider only one of them:

$$\sigma = \sum_{k_1=1}^{m_1} \cdots \sum_{k_s=1}^{m_s} \delta_p(k_1 + a k_2 + \cdots + a^{s-1} k_s).$$

By virtue of the consequence of Lemma 2, $\sigma > 0$ only when

$$m_1(m_2 \dots m_s)^{\alpha_1} > m_1(m_2 \dots m_s)^{\alpha_1 + \frac{1}{\alpha} - \frac{\alpha_1}{\alpha}} > q \geq c_2(\alpha, \alpha_1, s)\, p.$$

Define r, ρ, and n from the conditions:

$$m_1(m_2 \dots m_{r-1})^{\alpha_1} < q \leq m_1(m_2 \dots m_r)^{\alpha_1},$$

$$q = m_1(m_2 \dots m_{r-1}\rho)^{\alpha_1},$$

$$[\rho]\, n < m_r \leq [\rho]\, (n+1),$$

where n is an integer and $[\rho]$ is the integer part of ρ. Then

$$\sigma \leq \sum_{k_s=1}^{m_s} \cdots \sum_{k_{r+1}=1}^{m_{r+1}} \sum_{k_0'=0}^{n} \left[\sum_{k_r=k[\rho]+1}^{(k+1)[\rho]} \sum_{k_{r-1}=1}^{m_{r-1}} \cdots \sum_{k_1=1}^{m_1} \delta_p(k_1 + ak_2 + \dots + a^{s-1}k_s) \right].$$

The sum in square brackets does not exceed 1. Indeed, suppose it were greater than 1. This would mean that there exist at least two distinct solutions of the congruence $k_1 + ak_2 + \dots + a^{s-1}k_s \equiv 0 \pmod{p}$, i.e.

$$\delta_p(k_1' + \dots + a^r k_r' + a^{r+1}k_{r+1} + \dots + a^{s-1}k_s) = 1$$

and

$$\delta_p(k_1'' + \dots + a^r k_r'' + a^{r+1}k_{r+1} + \dots + a^{s-1}k_s) = 1.$$

In that case we would also have

$$\delta_p(k_1' - k_1'' + (k_2' - k_2'')a + \dots + (k_r' - k_r'')a^r) = 1.$$

But then

$$\overline{(k_1' - k_1'')\, [(k_2' - k_2'') \dots (k_r' - k_r'')]}^{\,\alpha_1} \leq m_1(m_2 \dots m_{r-1}\rho)^{\alpha_1} = q,$$

which contradicts the choice of a. Consequently,

$$\sigma \leq \sum_{k_s=1}^{m_s} \sum_{k_{r+1}=1}^{m_{r+1}} \sum_{k=0}^{n} 1 = m_s \dots m_{r+1}(n+1) \frac{(\rho m_{r-1} \dots m_2)^{\alpha_1} m_1}{q} \leq$$

$$\leq 4^{\alpha_1} \frac{m_1 (m_2 \dots m_s)^{\alpha_1}}{q}.$$

Hence, by virtue of (7) and (5), we obtain

$$|R| \leq c_3(\alpha, \alpha_1, s) \sum_{m_1 (m_2 \dots m_s)^{\alpha_1 + \frac{1}{\alpha} - \frac{\alpha_1}{\alpha}} > q} \frac{m_1 (m_2 \dots m_s)^{\alpha_1}}{q\, m_1^{\alpha+1} (m_2 \dots m_s)^{\alpha \alpha_1 + 1}} \leq c(\alpha, \alpha_1, s) p^{-\alpha}.$$

The theorem is proved.

In Theorem 1 it was assumed that the number of nodes of the quadrature formula coincides with the quantity p, which determined the form of the grid. Thus, when the number of nodes was changed, all nodes changed. In contrast, in the following theorem we consider such quadrature formulas in which increasing the number of nodes by one leads only to the addition of one new node to the old ones. In lectures by N. N. Chentsov at Moscow University it was proved that, for the error of quadrature formulas of this type, one cannot obtain an estimate better than $R = O(N^{-1})$, where N is the number of nodes.

Theorem 2. Let a be chosen as in Lemma 2; let $N > 0$ be an arbitrary integer not exceeding p. For functions $f \in E_{s-1}^{\alpha}$ the quadrature formula

$$\int_0^1 \cdots \int_0^1 f(x_1, \dots, x_{s-1})\, dx_1 \dots dx_{s-1} - \frac{1}{N} \sum_{k=1}^{N} f\left(\frac{a}{p} k, \dots, \frac{a^{s-1}}{p} k\right) = R, \qquad (8)$$

where

$$|R| \leq \frac{c_4(\alpha, s)}{N}.$$

Proof. According to (6),

$$|R| \ll \frac{1}{N} \sum_{m_1, \dots, m_{s-1} = -\infty}^{\infty}{}' (\overline{m}_1 \dots \overline{m}_{s-1})^{-\alpha} \left| \sum_{k=1}^{N} \exp\left[\frac{a m_1 + \dots + a^{s-1} m_{s-1}}{p} k \right] \right|.$$

Using the well-known estimate of a trigonometric sum, we obtain

$$|R| \ll \frac{2}{N} \sum_{m_1, \dots, m_{s-1} = -\infty}^{\infty}{}' (\overline{m}_1, \dots, \overline{m}_{s-1})^{-\alpha} \min\left(N, \frac{1}{\left\langle\!\left\langle \frac{a m_1 + \dots + a^{s-1} m_{s-1}}{p} \right\rangle\!\right\rangle} \right), \qquad (9)$$

where $\langle\langle\alpha\rangle\rangle$ is the distance from α to the nearest integer. There always exists such an m_0 that $am_1 + \ldots + a^{s-1}m_{s-1} \equiv m_0 \pmod p$ and $|m_0| \le \frac{1}{2}(p-1) = p_1$. Obviously, the following equality holds:

$$\left\langle\left\langle \frac{am_1 + \ldots + a^{s-1}m_{s-1}}{p} \right\rangle\right\rangle^{-1} = \sum_{m_0=-p_1}^{p_1} \left\langle\left\langle \frac{m_0}{p} \right\rangle\right\rangle^{-1} \delta_p(am_1 + \ldots + a^{s-1}m_{s-1} - m_0).$$

Since

$$\left\langle\left\langle \frac{m_0}{p} \right\rangle\right\rangle = \frac{|m_0|}{p},$$

it follows from (9) that

$$\begin{aligned} |R| &\ll \frac{2p}{N} \sum_{m_1,\ldots,m_{s-1}=-\infty}^{\infty}{}' \sum_{m_0=-p_1}^{p_1} \frac{\delta_p(am_1 + \ldots + a^{s-1}m_{s-1} - m_0)}{m_0(\overline{m}_1 \ldots \overline{m}_{s-1})^{\alpha}} \\ &\ll \frac{p^{1+\varepsilon}}{N} \sum_{m_1,\ldots,m_{s-1}=-\infty}^{\infty}{}' \sum_{m_0=-p_1}^{p_1} \frac{\delta_p(am_1 + \ldots + a^{s-1}m_{s-1} - m_0)}{m_0^{1+\varepsilon}(\overline{m}_1 \ldots \overline{m}_{s-1})^{\alpha}}, \end{aligned} \tag{10}$$

where $0 < \varepsilon < \alpha - 1$. Applying the result of Theorem 1 to (10), we obtain assertion (8).

Remark. The nodes obtained in Theorem 2 can be used to construct quadrature formulas with different weights for $N \le p$. Namely, by the method of paper [3], for functions of the class E_s^{α}, where α is not an integer, one can construct such formulas whose error has order $N^{-[\alpha]}$.

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Note: Figure translations are in progress. See original paper for figures.