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Abstract

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ASYMPTOTIC EXPANSIONS FOR WEAK INTERACTION IN A NEIGHBORHOOD OF THE CRITICAL TEMPERATURE OF A PHASE TRANSITION IN A MODIFIED FORMULATION OF THE PROBLEM OF A NONIDEAL BOSE-EINSTEIN SYSTEM

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Earlier, in ([1, 2]), a method was proposed for constructing asymptotic expansions, for weak interaction, in a modified formulation of the problem of a nonideal Bose-Einstein system. This method is based on the problem of finding, for small interaction, an asymptotic solution of a system of nonlinear integral equations for $A(p)$, $B(p)$, and $\varepsilon(p)$ (see (1), (2), (3) in ([2])):

$$A(p) = E(p) + n_0 v(p) + \frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2 dp' \left(v(p,p') - v(p')\right)\left(\frac{A(p')}{\varepsilon(p')}\,\mathrm{cth}\,\frac{\beta\varepsilon(p')}{2} - 1\right) + \frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2 dp'\, v(p')\, \frac{B}{\varepsilon} \tag{1}$$

$$B(p) = n_0 v(p) - \frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2 dp'\, v(p,p')\,\frac{B(p')}{\varepsilon(p')}\,\mathrm{cth}\,\frac{\beta\varepsilon(p')}{2}, \tag{2}$$

$$\varepsilon^2(p) = A^2(p) - B^2(p), \tag{3}$$

where n_0 is the density of the number of particles in the condensate, $\beta = 1/\theta$, $\theta = kT$, k is Boltzmann' s constant, and T is the absolute temperature.

In ([2, 3]) results are given that were obtained with the aid of the asymptotic solution, constructed by us at that time, of (1), (2), (3) in the form of expansions for small v and fixed n_0 for the case $\theta = 0$ and for the case $\theta \neq 0$. (Because of the nonuniform convergence of these expansions in the temperature region $\theta = 0$, it was necessary to study this case separately.) Expansions of the indicated type turn out to be unsuitable, as was already noted in ([2]), for studying the situation near the critical temperature of the phase transition, since they converge nonuniformly in the region $n_0 = 0$.

At that time we tried to get around the indicated difficulty by means of an asymptotic solution of (1), (2), (3) in the form of expansions for small n_0 and fixed v, in order to study a neighborhood of the critical temperature (see (14) in ([3])). However, expansions of this kind concealed another difficulty. They turned out to be nonuniformly convergent in the region $v = 0$. Of course, this is very bad, since the equations (1), (2), (3) themselves are guaranteed only in the limit of small v, as follows from ([1]). In the present note we give the results of studying, with the aid of (1), (2), (3), the region of the critical temperature of the phase transition.

Simultaneously with the small interaction, we shall assume that the temperature is close to the phase-transition temperature θ_0 of the ideal Bose–Einstein system, and also that n_0 is close to zero. More precisely, we set

$$\theta = \theta_0 - nv(0)\delta$$

and

$$n_0 = \varkappa s_0,$$

where

$$\varkappa = \frac{m^3 v(0)\theta_0^2}{\hbar^6},$$

and we fix δ and s_0 (instead of fixing θ and n_0).

If we now seek the solutions of (1), (2), (3) in the form of formal expansions in integral powers of v, then the corrections v and v^2 for A, B will be quite reasonable, but the correction v^3 will contain a divergent integral at small momenta. This means, as it turns out, that in the actual expansion—

there are additional asymptotic terms of order v^2, which cannot be obtained by the indicated formal procedure.

Thus, we can write

$$\begin{aligned} A(p) = E(p) &+ \frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2 dp' \left(v(p,p') - v(p')\right)\left(\operatorname{cth}\frac{E(p')}{2\theta_0} - 1\right) + s_0\varkappa v(p) \\ &- \frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2 dp' \left(v(p,p') - v(p')\right)\frac{1}{\operatorname{sh}^2 E(p')/2\theta_0}\left(\frac{\varepsilon_1(p')}{2\theta_0} + \frac{E(p')}{2\theta_0^2}nv(0)\delta\right) \\ &+ \mathfrak{A}(p) + \text{terms higher than } v^2, \end{aligned} \tag{4}$$

$$B(p) = s_0\varkappa v(p) + \mathfrak{B}(p) + \text{terms higher than } v^2, \tag{5}$$

where $\mathfrak{A}(p)$, $\mathfrak{B}(p)$ are as yet unknown second-order terms, which arise from the correct asymptotics of the correction terms if, in the exact equations (1), (2), (3), one separates out the formal first- and second-order terms written in (4) and (5).

In order to calculate these correct asymptotics, it is necessary to know the expansions of the stretched solutions A and B. We shall stretch the momentum $p = \rho x$, $\rho = 2m^{1/2}s_0^{1/2}\varkappa^{1/2}v^{1/2}(0)$. From (4), (5) we obtain

$$A(\rho x) = 2s_0\varkappa v(0)a(x), \quad B(\rho x) = 2s_0\varkappa v(0)b(x), \quad \varepsilon(\rho x) = 2s_0\varkappa v(0)e(x), \tag{6}$$

where

$$\begin{gathered} a(x) = x^2 + \frac{1}{2} + \mathfrak{A}, \qquad b(x) = \frac{1}{2} + \mathfrak{B}, \\ e^2(x) = \left(x^2 + \frac{1}{2} + \mathfrak{A}\right)^2 - \left(\frac{1}{2} + \mathfrak{B}\right)^2, \\ \mathfrak{A}(0) = 2s_0\varkappa v(0)\mathfrak{A}, \qquad \mathfrak{B}(0) = 2s_0\varkappa v(0)\mathfrak{B}. \end{gathered} \tag{7}$$

We shall now obtain the asymptotics for $\mathfrak{A}(p)$ and $\mathfrak{B}(p)$ if, in the exact correction terms in (4), (5) (which cannot be formally expanded in v), we stretch $p = \rho x$ in the region of integration near $p = 0$. Then we obtain expressions for $\mathfrak{A}(p)$, $\mathfrak{B}(p)$ in terms of $a(x)$, $b(x)$, $e(x)$. In fact, these are equations for $\mathfrak{A}$, $\mathfrak{B}$. From them we have $\mathfrak{B} = -\mathfrak{A}$, $\mathfrak{A} = \frac{1}{2} - \frac{1}{2}b$, and for b we have the equation

$$\frac{1}{b} = 1 + \frac{1}{2\pi s_0^{1/2} b}\left(1 - \sqrt{1-b}\right). \tag{8}$$

With the aid of the expansions for A, B one can obtain expansions for $n(s_0, \delta)$ and $\mu(s_0, \delta)$ (see (4), (5) from (2)). Next, with the aid of an approximate solution of the equation $n(s_0, \delta) = n$, where the density of the system is fixed, we obtain

$$s_0 = \frac{1}{64\pi^2}\frac{n^2\hbar^6}{m^6\theta_0^3}\left(\int_0^{+\infty} p'^2\,dp'\,\frac{E(p')}{\operatorname{sh}^2 E(p')/2\theta_0}\right)^2(\delta - \delta_0)^2 + \cdots, \tag{9}$$

where δ_0 determines the shift of the critical temperature for a nonideal system

$$\begin{aligned} \delta_0 = -\frac{\theta_0}{nv(0)}\frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2\,dp'\,\frac{1}{\operatorname{sh}^2 E(p')/2\theta_0}\int_0^{+\infty} p''^2\,dp''\,(v(p', p'') - v(p'')) \\ \times\left(\operatorname{cth}\frac{E(p'')}{2\theta_0} - 1\right)\bigg/\int_0^{+\infty} p'^2\,dp'\,\frac{E(p')}{\operatorname{sh}^2 E(p')/2\theta_0}. \end{aligned} \tag{10}$$

Next, substituting into the expansion for $\mu(s_0, \delta)$ the expansion for the function $s_0(\delta)$ found above, we obtain

$$\begin{aligned}\mu(\delta) = {} & \frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2\,dp'\,(v(0)+v(p'))\left(\operatorname{cth}\frac{E(p')}{2\theta_0}-1\right) \\ & -\frac{1}{8\pi^2\hbar^3\theta_0}\int_0^{+\infty} p'^2dp'\,(v(0)+v(p'))\frac{1}{\operatorname{sh}^2 E(p')/2\theta_0}\frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p''^2dp''\,(v(p,p')-v(p''))\times \\ & \times\left(\operatorname{cth}\frac{E(p'')}{2\theta_0}-1\right)-\frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2\,dp'\,(v(0)+v(p'))\frac{1}{\operatorname{sh}^2 E(p')/2\theta_0}\frac{E(p')}{2\theta_0^3}nv(0)\delta \\ & -\frac{1}{4\pi^2\hbar^3}\frac{nv^2(0)}{\theta_0^2}\int_0^{+\infty} p'^2\,dp'\,\frac{E(p')}{\operatorname{sh}^2 E(p')/2\theta_0}(\delta_0-\delta) \\ & -\frac{1}{32\pi^2}\frac{n^2v^2(0)}{m^3\theta_0^6}\left(\int_0^{+\infty} p'^2\,dp'\,\frac{E(p')}{\operatorname{sh}^2 E(p')/2\theta_0}\right)^2(\delta_0-\delta)^2 \\ & +\text{terms of order } v^2 \text{ and higher } (\delta_0-\delta)^2+\text{terms higher than } v^2. \end{aligned} \tag{11}$$

The terms of order v^2 are not guaranteed, since for a correct calculation of these terms it is also necessary to take into account the contribution from second-order diagrams (see (?))*.

For temperatures above the critical one we must consider the following equations:

$$\varepsilon(p) = E(p)-\mu+\frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2dp'\,(v(0)+v(p,p'))\left(\operatorname{cth}\frac{\beta\varepsilon(p')}{2}-1\right), \tag{12}$$

$$n = \frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2dp'\left(\operatorname{cth}\frac{\beta\varepsilon(p')}{2}-1\right), \tag{13}$$

in which the density n, and not the chemical potential μ, is fixed.

Simultaneously with the small interaction we shall assume that the temperature is close to the phase-transition temperature θ_0, and also that μ is small. More precisely, set $\theta = \theta_0 - nv(0)\delta$ and

$$\mu = \frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2dp'\,(v(0)+v(p'))\left(\operatorname{cth}\frac{E(p')}{2\theta_0}-1\right)+2nv(0)\chi\gamma$$

and we shall regard δ and γ as fixed (instead of fixing θ and μ).

If we now seek the solution of (12) in the form of a formal expansion in powers of v, then the corrections v and v^2 will be reasonable, but the correction v^3 will contain an integral divergent at small momenta. This means, it turns out, that in the actual expansion there are additional asymptotic terms of order v^2, whose reflection is the divergent terms in the formal expansion.

Thus, we have

$$\varepsilon(p) = E(p) + \varepsilon_1(p) - 2nv(0)\chi\gamma - \frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2 dp' \, (v(0) + v(p,p'))\times \\ \times \frac{1}{\operatorname{sh}^2 E(p')/2\theta_0}\left(\frac{\varepsilon_1(p')}{2\theta_0} + \frac{E(p')nv(0)}{2\theta_0^2}\delta\right) + \mathfrak{C}(p) + \text{terms higher than } v^2, \tag{14}$$

where $\mathfrak{C}(p)$ are as yet unknown second-order terms, similar to $\mathfrak{A}(p)$, $\mathfrak{B}(p)$.

Let us find the stretched solution. Let $p = \rho x$, $\rho = 2m^{1/2}n^{1/2}\chi^{1/2}v^{1/2}(0)$. We obtain from (14) that

$$\varepsilon(\rho x) = 2nv(0)\chi e(x), \qquad \mathfrak{C}(0) = 2nv(0)\chi\mathfrak{C},$$

$$e(x) = x^2 + \mathfrak{C} - \gamma - \frac{1}{8\pi^2\hbar^3\, nv(0)\chi}\int_0^{+\infty} p'^2 dp' \, (v(0) + v(p'))\,\frac{1}{\operatorname{sh}^2 E(p')/2\theta_0}\times \\ \times\left(\frac{\varepsilon_1(p')}{2\theta_0} + \frac{E(p')nv(0)}{2\theta_0^2}\delta\right). \tag{15}$$

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* The contribution from second-order diagrams is easy to take into account. It does not depend on δ. See below.

We now use the stretched solution found above in order to determine the asymptotics of the correction integral that is obtained if, in the exact formula (12), the terms v and v^2 written explicitly in (14) are separated off. Then for $\mathcal{E}$ we obtain the equation

$$\mathcal{E} = -\frac{1}{\pi\hbar^3}\frac{m^{1/2}v^{1/2}(0)\theta_0}{n^{1/2}\chi^{1/2}}\left(\mathcal{E} - \gamma - \frac{1}{8\pi^2\hbar^3}\frac{1}{nv(0)\chi}\int_0^{+\infty} p'^2\, dp' \, (v(0) + v(p'))\frac{1}{\operatorname{sh}^2 E(p')/2\theta_0}\left(\frac{\varepsilon_1(p')}{2\theta_0} + \frac{E(p')nv}{2\theta_0^2}\right.\right. \tag{16}$$

Now we use (13). From it we find

$$-\frac{1}{4\pi^2\hbar^3}\int_0^{+\infty} p'^2\,dp'\,\frac{1}{\operatorname{sh}^2 E(p')/2\theta_0}\frac{E(p')nv(0)}{2\theta_0^2}(\delta-\delta_0)-\frac{1}{\pi\hbar^3}m^{3/2}n^{1/2}\chi^{1/2}v^{1/2}(0)\theta_0\times$$
$$\times\left(\mathcal{E}-\gamma-\frac{1}{8\pi^2\hbar^3}\frac{1}{nv(0)\chi}\int_0^{+\infty} p'^2\,dp'\,(v(0)+v(p'))\frac{1}{\operatorname{sh}^2 E(p')/2\theta_0}\left(\frac{\varepsilon_1(p')}{2\theta_0}+\frac{E(p')nv(0)}{2\theta_0^2}\,\delta\right)\right)^{1/2}$$
$$+\text{ terms higher than } v^2=0, \tag{17}$$

where the expression for δ_0 is given by (10); δ_0 gives the shift of the critical temperature due to the interaction. Solving (16) and (17) simultaneously, we obtain an expression for the chemical potential that coincides completely with (11).

Thus, the critical temperature θ_c, calculated from the side of high and from the side of low temperatures, proves to be the same. At the critical point, both the chemical potential itself and its first and second derivatives with respect to temperature prove to be continuous.

For lack of space we do not give the results of the calculation of the free energy Ψ_0/V in the region of critical temperatures. A calculation was carried out for $\theta > \theta_0$ and a calculation for $\theta < \theta_0$. At $\delta = \delta_0$, the free energy itself and its first and second derivatives prove to be continuous. For the heat capacity one obtains the expression for an ideal Bose-Einstein system at $\theta = \theta_0$.

Let us now note what is given by taking into account correction diagrams of the second and higher orders (see ([2])). In the asymptotic terms v^2, the only correction arises from second-order diagrams without n_0. Their contribution does not depend on δ. Second-order terms with n_0 will lead to a contribution of the type $s_0 v^3 \ln v s_0$ (see (15) in ([2])).

It is necessary to calculate the next asymptotic order in Ψ/V in order to draw a conclusion about the temperature dependence of the heat capacity near the critical temperature of the phase transition.

The next order after v^2 is an order of the type $v^3 \ln v$. A detailed analysis shows that every correction diagram of any order gives a contribution to the asymptotic order v^3.

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