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Abstract

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ON THE CALCULATION OF STATIONARY RESONANT REGIMES OF CERTAIN NONLINEAR OSCILLATORY SYSTEMS

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§ 1. **Stationary resonant regimes.** Let a system with one degree of freedom, describing oscillatory or rotational motions, have the form

$$\frac{d}{dt}\left[m(\mu)\dot{y}\right] + Q(\mu, y) = \varepsilon f(\mu, y, \dot{y}, \vartheta, \varepsilon),$$

$$\dot{\vartheta} = \nu(\mu) + \varepsilon\Theta(\mu, y, \dot{y}, \vartheta, \varepsilon), \qquad \dot{\mu} = \varepsilon M(\mu, y, \dot{y}, \vartheta, \varepsilon), \tag{1}$$

where y is a one-dimensional coordinate; $m(\mu)$ is the mass; $\varepsilon > 0$ is a small parameter; $\mu = (\mu_1, \dots, \mu_l)$ is a collection of slowly varying parameters; $Q(\mu, y)$ is a potential force causing oscillation or rotation; f is a small nonlinear perturbation; ϑ is a function describing the dependence of the perturbation on time. It is assumed that, in the case of rotation, all the functions entering (1) depend periodically on y with period 2π, and Q has zero mean value with respect to y (when studying oscillatory regimes these conditions may be omitted); moreover, both for oscillatory and for rotational regimes all functions in (1) are periodic in ϑ with period 2π. Some problems leading to systems that are special cases of (1) were considered in ([1−5]).

By a certain change of variables (for systems that are a special case of (1), similar transformations were performed in ([1−5])), system (1) can be reduced to the form

$$\dot{c} = \varepsilon C(c, \mu, \psi, \vartheta, \varepsilon),$$

$$\dot{\mu} = \varepsilon M(c, \mu, \psi, \vartheta, \varepsilon),$$

$$\dot{\psi} = \omega + \varepsilon\Psi(c, \mu, \psi, \vartheta, \varepsilon),$$

$$\dot{\vartheta} = \nu + \varepsilon\Theta(c, \mu, \psi, \vartheta, \varepsilon), \tag{2}$$

where ψ is the phase of the fundamental oscillations or rotations; the vector c is a certain collection of slowly varying parameters characterizing the oscillations or rotations (for example, the amplitude of oscillations, energy, action, etc.); ω is the natural frequency of the system in the zero approximation (for $\varepsilon = 0$), depending, generally speaking, on c and μ. The functions C, M, Ψ, Θ in system (2) are periodic in c and μ. If, for certain values of c, μ, the equality

$$p\omega = q\nu, \tag{3}$$

holds, where p and q are relatively prime integers, then we say that resonance occurs in the system. In the present paper the question is considered of the existence of stable (in a certain sense) resonant regimes of system (1) in a neighborhood of resonance points.

§ 2. **Statement of the problem.** We shall consider systems of type (2) independently of the original equations (1). For generality let us assume that the frequency ν depends not only on the variables μ, but also on c (as does ω). Consider the resonance determined by the numbers p and q. Combining the slowly varying variables c and μ into the vector $x = (x_1, \dots, x_n)$ and passing from

variables ψ and ϑ to $\varphi = \vartheta - \frac{p}{q}\psi,\ \beta = \frac{1}{q}\psi$, we rewrite system (2) in the following form:

$$\begin{aligned} \dot{x} &= \varepsilon X(x, \varphi, \beta, \varepsilon), \\ \dot{\varphi} &= \lambda(x) + \varepsilon\Phi(x, \varphi, \beta, \varepsilon), \\ \dot{\beta} &= \Omega(x) + \varepsilon B(x, \varphi, \beta, \varepsilon), \end{aligned} \tag{4}$$

where $\lambda(x) = \nu(x) - \frac{p}{q}\omega(x)$, and $\lambda(x_0) = 0$ (x_0 is a root of equation (3)). The functions X, Φ, B are periodic in φ and β with period 2π. The variable φ characterizes the phase detuning between the external action and the system' s proper oscillations. Asymptotic methods based on averaging have been applied to systems similar to (4) in a number of works (for example, ([1−5])). In some cases (this is noted, for example, in ([1])) the usual scheme of the averaging method, when applied to system (4), leads to the appearance of everywhere discontinuous functions and of divergent series (because of the so-called small denominators).

In the present work a specialized averaging scheme was used, in which there are no series with small denominators. For this purpose, the averaging scheme is constructed in such a way that the coefficients of the transformation of system

(4) to the averaged system (the latter is not written out here) contain only the fixed values $x = x_0$ corresponding to the resonance point. The coordinates of the resonance point (in the zero approximation, i.e., for $\varepsilon = 0$), as usual, are determined from the condition that the mean rates of change of the quantities x and φ vanish:

$$\overline{X}(x_0, \varphi_0) = 0, \qquad \lambda(x_0) = 0,$$

where the bar denotes averaging with respect to β for $\varepsilon = 0$. It is assumed that such values x_0, φ_0 exist. The problem consists in clarifying the conditions under which, in a neighborhood of the resonance point x_0, φ_0, there exist stable stationary regimes, i.e., such solutions of system (4) which, if at the initial moment of time they are sufficiently close to the resonance point, remain close to it for arbitrarily large time intervals for sufficiently small values of ε.

§ 3. **Main results.** We introduce the characteristic equation as follows:

$$\det(A - kE) = 0, \tag{5}$$

where

$$A = \begin{pmatrix} \varepsilon \dfrac{\partial \overline{X}(x_0, \varphi_0)}{\partial x} & \varepsilon \dfrac{\partial \overline{X}(x_0, \varphi_0)}{\partial \varphi} \\ \dfrac{\partial \lambda(x_0)}{\partial x} + \varepsilon \dfrac{\partial \overline{\Phi}(x_0, \varphi_0)}{\partial x} + \dfrac{\partial^2 \lambda(x_0)}{\partial x^2} \varepsilon \delta x & \varepsilon \dfrac{\partial \overline{\Phi}(x_0, \varphi_0)}{\partial \varphi} \end{pmatrix},$$

$\varepsilon \delta x$ is the first-order correction to the coordinates of the resonance point (we do not write out here the formulas for computing these corrections), and E is the identity matrix of the corresponding dimension. The expressions $\partial\lambda/\partial x$, $\partial^2\lambda/\partial x^2$, $\partial\overline{X}/\partial x$, etc. denote here vectors and matrices of the corresponding dimensions.

We require that all roots of equation (5) have negative real parts. Under this basic condition and certain other restrictions it is shown that, for arbitrarily large $T > 0$ and arbitrarily small $\xi > 0$, there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ there exists $\delta(\varepsilon)$ such that from the condition $\max |x(t_0) - x_0, \varphi(t_0) - \varphi_0| < \delta$ for all $t_0 \le t \le T$

there follows the inequality $\max |x(t) - x_0,\ \varphi(t) - \varphi_0| < \xi$, where $x(t)$, $\varphi(t)$ are a solution of system (4). It can also be shown that, for sufficiently small values of ε, on a time interval of order $\varepsilon^{-1/2}$ the solutions of system (4) do not leave a certain neighborhood of the resonance point of size of order $\varepsilon^{1/2}$, provided that the initial values for system (4) are specified in some ε-neighborhood of the resonance point. In proving these assertions one uses the specialized averaging scheme discussed in § 2.

§ 4. **Applications.** In papers (6–12), formulas were obtained for the nonresonant regimes of system (1) which make it possible to find oscillatory or rotational motions with any degree of accuracy. The results of § 3 make it possible to find analogous expressions also in the resonant case. It is essential that, in order to use these expressions, it is enough to know the coefficients of the original system (1), and no preliminary computation is required of the solution of the unperturbed system into which (1) passes when $\varepsilon = 0$.

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