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1963

Abstract

Full Text

CONTINUUM MECHANICS

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ON THE THEORY OF COMPLEX MEDIA

(Presented by Academician A. Yu. Ishlinskii, July 4, 1962)

The paper considers questions of constructing a relation between the stress and strain states for a sufficiently broad class of models of continuous media.

1. We shall base the constructions on three fundamental mechanisms of deformation: elastic, viscous, and plastic. The first mechanism determines a reversible deformation process; the latter two, an irreversible one. To illustrate the properties of complex media we shall use dynamical models ([1–3]). In such models force corresponds to stresses, and displacements to strains of the medium being modeled.

Fig. 1

The plastic mechanism P—the mechanism of dry friction—is shown in Fig. 1a, and the viscous mechanism V in Fig. 1b, where σ is tensile stress, ε is strain, and t is time. A characteristic feature of the mechanisms P and V is the one-sided application of the external force. For the elastic mechanism E (Fig. 1c), it is necessary to attach one end of the spring to a rigid wall. The rigid wall may be interpreted as a dry-friction mechanism with an arbitrarily large coefficient of adhesion.

Let us consider the simplest combinations of mechanisms: EP, a model of an elastoplastic body (Fig. 2a), and EV, a model of a viscoelastic body (Fig. 2b). For these models the total strain is composed of elastic and plastic, or elastic and viscous, strain:

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p, \qquad d\varepsilon = d\varepsilon^e + d\varepsilon^v. \tag{1,1}$$

In the case of a sequential action of the mechanisms P and E (Fig. 2c) or V and E (Fig. 2d), the corresponding deformation is only plastic or viscous; therefore we shall denote these mechanisms, respectively, by Pe and Ve. In this case the elastic elements, being internal, do not change the nature of the deformation, while substantially affecting the character of the σ-ε dependence.

Thus, capital letters indicate the mechanism that determines the character of the deformation; internal mechanisms that do not change the character of the deformation will be denoted, respectively, by lowercase letters. We note that in the case under consideration the deformation may have the following character:

Fig. 2

Figure 1: Fig. 2

Fig. 3

Figure 2: Fig. 3

elastic, viscous, plastic, viscoelastic, elastoplastic. A model consisting of the mechanisms P and V should be denoted by Pv, since the deformation is plastic in character and the element V in this case is internal (Fig. 2e). To verify this, it is sufficient to unload the system. Similarly, a mechanism consisting of the elements V and P should be denoted by Vp (Fig. 2f).

From the foregoing, the proposed principle for indexing models is clear. For example, the model shown in Fig. 3a should be denoted $EPevepe$; the model shown in Fig. 3b, $Vepvep$.

2. To construct the relation between the stress tensor σ_{ij} and the strain tensor ε_{ij}, following the ideas of works ([4,5]), let us consider two-dimensional dynamic models. In Fig. 2 a'—zh' are shown two-dimensional dynamic models corresponding to the one-dimensional models presented in Fig. 2 a—zh.

Fig. 2

In Fig. 4a a two-dimensional dynamic model $EVeveve$ is shown, or, as it is more convenient to denote it, $EVe_1v_1e_2v_2e_3$. Obviously,

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{v}. \tag{2.1}$$

Assigning henceforth an overbar to the deviators of the corresponding tensors and, for simplicity, everywhere taking the material to be incompressible (moreover, we shall regard all tensors of both actual and internal deformations as deviators), we obtain

$$d\varepsilon_{ij}^{e} = \frac{1}{2G}\,\sigma'_{ij}, \tag{2.2}$$

where G is the shear modulus.

Let us denote the tensor of internal stresses (microstresses, according to the terminology of work ([5])) corresponding to the element e_n by $s_{ij}^{\prime(n)}$; the tensor of internal strains (microstrains) corresponding to the displacements of the element v_n, by $\chi_{ij}^{(n)}$. We obtain

Fig. 3

Fig. 4

Fig. 4

Figure 3: Fig. 4

$$d\varepsilon_{ij}^{v} = \mu\, dt \left(\sigma'_{ij} - s_{ij}^{\prime(1)}\right), \qquad \overline{d\chi_{ij}^{(1)}} = \mu_1\, dt \left(s_{ij}^{\prime(1)} - s_{ij}^{\prime(2)}\right),$$

$$d\chi_{ij}^{(2)} = \mu_2\, dt \left(s_{ij}^{\prime(2)} - s_{ij}^{\prime(3)}\right), \tag{2.3}$$

where $\mu,\ \mu_n$ are the viscosity coefficients of the elements $V,\ v_n$.

To relations (2,3) one should add the conditions determining the stresses corresponding to the tensions in the elements e_n:

$$d\varepsilon_{ij}^{v} - d\chi_{ij}^{(1)} = c_1\, ds_{ij}^{\prime(1)}, \qquad d\chi_{ij}^{(1)} - d\chi_{ij}^{(2)} = c_2\, ds_{ij}^{\prime(2)}, \qquad d\chi_{ij}^{(3)} = c_3\, ds_{ij}^{\prime(3)}, \ (2,4)$$

where $1/c_n$ are the stiffness coefficients of the elements e_n.

In Fig. 4 a two-dimensional dynamic model $EPe_1p_1e_2p_2e_3$ is shown. Obviously,

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p}, \tag{2,5}$$

and relation (2,2) is valid.

The plasticity condition corresponding to the element P will be written in the form $f(\sigma_{ij} - s_{ij}^{(1)}) = k$; the plasticity condition corresponding to the element p_n, in the form $\varphi_n(s_{ij}^{(n)} - s_{ij}^{(n+1)}) = k_n$. The tensor of internal strains (microstrains), corresponding to the displacements of the elements p_n, will be denoted by $\chi_{ij}^{(n)}$. We shall have

$$f(\sigma_{ij} - s_{ij}^{(1)}) = k, \qquad \varphi_1(s_{ij}^{(1)} - s_{ij}^{(2)}) = k_1, \qquad \varphi_2(s_{ij}^{(2)} - s_{ij}^{(3)}) = k_2, \tag{2,6}$$

$$d\varepsilon_{ij}^{p} = d\lambda\, \frac{\partial f}{\partial \sigma_{ij}}, \qquad d\chi_{ij}^{(1)} = d\lambda_1\, \frac{\partial \varphi_1}{\partial s_{ij}^{(1)}}, \qquad d\chi_{ij}^{(2)} = d\lambda_2\, \frac{\partial \varphi_2}{\partial s_{ij}^{(2)}}, \tag{2,7}$$

$$d\varepsilon_{ij}^{v} - d\chi_{ij}^{(1)} = c_1\, ds_{ij}^{\prime(1)}, \qquad d\chi_{ij}^{(1)} - d\chi_{ij}^{(2)} = c_2\, ds_{ij}^{\prime(2)}, \qquad d\chi_{ij}^{(2)} = c_3\, ds_{ij}^{\prime(3)}, \ (2,8)$$

where $d\lambda,\ d\lambda_n$ are proportionality coefficients.

In order that the strain tensors be deviators, it is necessary to assume that all the plasticity conditions do not depend on the first invariants of the corresponding tensors.

In relations (2,7) the associated flow law is used. The representations of the associated flow law follow from extremum considerations for the increment of work of stresses on the corresponding strain increments. These considerations were used above both for actual and for internal stresses and strains. The differentiation in (2,7) is carried out with respect to the first of the stresses, although the differentiation may be carried out with respect to the total stresses, since

$$\frac{\partial(\sigma_{ij} - s_{ij}^{(1)})}{\partial\sigma_{ij}} = 1, \qquad \frac{\partial(s_{ij}^{(n)} - s_{ij}^{(n+1)})}{\partial s_{ij}^{(n)}} = 1.$$

Therefore

$$\frac{\partial f}{\partial\sigma_{ij}} = \frac{\partial f}{\partial(\sigma_{ij} - s_{ij}^{(1)})}, \qquad \frac{\partial\varphi_n}{\partial s_{ij}^{(n)}} = \frac{\partial\varphi_n}{\partial(s_{ij}^{(n)} - s_{ij}^{(n+1)})}.$$

The construction of the relation $\sigma_{ij} - \varepsilon_{ij}$ for media including both composite elements $v_n,\ p_m$ presents no difficulty. For example, for the model EPe_1ve_2p the relations of the sought connection are written in the form (2,5), (2,2); furthermore, one will also have

$$f(\sigma_{ij} - s_{ij}^{(1)}) = k, \qquad \varphi(s_{ij}^{(2)}) = k_1; \tag{2,9}$$

$$d\varepsilon_{ij}^p = d\lambda\,\frac{\partial f}{\partial\sigma_{ij}}, \qquad d\chi_{ij} = \mu\,dt\,(s_{ij}^{(1)} - s_{ij}^{(2)}), \qquad d\chi_{ij} = d\lambda_1\,\frac{\partial\varphi}{\partial s_{ij}^{(2)}}; \tag{2,10}$$

$$d\varepsilon_{ij}^p - d\chi_{ij} = c_1\,ds_{ij}^{(1)}, \qquad d\chi_{ij} - d\chi_{ij} = c_2\,ds_{ij}^{(2)}. \tag{2,11}$$

In work ([6]) one case of a similar relation was considered.

Let us make several remarks. The approach set forth to the construction of the relation $\sigma_{ij} - \varepsilon_{ij}$ is a direct generalization of the approach developed in the theory of translational hardening. In the present case, not only the principal but also the internal mechanisms of plasticity and viscosity define their own loading surfaces, which undergo translation in their stress spaces.

The case in which the bond between individual elements is rigid is interpreted as the limiting case considered when the corresponding coefficients c_n tend to zero.

Nonlinear effects can be taken into account if one assumes that the quantities k_n, μ_n, c_n depend on the invariants of the corresponding tensors.

It presents no fundamental difficulties to use piecewise linear potential surfaces ([7–10]).

The relations considered establish a connection between the deviators of the corresponding tensors. The dependence between the first invariants of the corresponding tensors, which determines the compressibility of the material, can be established independently ([11]).

Let us note that the model EV corresponds to the Maxwell body, and the model Ve to the Voigt body ([1–3]). Usually in the literature a two-sided application of the external force is considered, and for the Voigt model the elements E and V are connected in parallel. Such a schematization is inconvenient in constructing the corresponding two-dimensional models.

As one of the basic mechanisms one may also use the mechanism of ideal hardening ([12,13]), which may make it possible to take into account a number of new interesting effects.

The author expresses deep gratitude to Academician L. I. Sedov for a number of valuable comments.

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Received
29 IV 1962

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Note: Figure translations are in progress. See original paper for figures.