Reports of the Academy of Sciences of the USSR

N. N. KOCHINA

1963

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1963. Vol. 148, No. 1

HYDROMECHANICS

N. N. KOCHINA

ON THE DYNAMIC EXPLOSION OF EQUILIBRIUM IN CERTAIN IDEAL MEDIA AT ZERO TEMPERATURE GRADIENT

(Presented by Academician L. I. Sedov on 5 VII 1962)

The problem of the dynamic explosion of equilibrium in an ideal gravitating gas for adiabatic motion with adiabatic exponent $\gamma = 7/6$ was investigated by L. I. Sedov (^{1}). The problem of the dynamic explosion of equilibrium in a gas with gravitation, under the assumption that in the region of disturbed motion the gas temperature depends only on time, was considered by E. V. Ryazanov (^{2}). Below we consider the problem of the dynamic explosion of equilibrium in a certain ideal medium at zero temperature gradient.

In the disturbed region there hold the equations of one-dimensional unsteady motion with spherical symmetry, with zero temperature gradient and with gravitation:

$$\frac{\partial M}{\partial r} = 4\pi r^2 \rho; \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + \frac{2\rho v}{r} = 0; \tag{2}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{GM}{r^2} = 0; \tag{3}$$

$$\frac{\partial T}{\partial r} = 0. \tag{4}$$

Here M is the mass of the medium inside a sphere of radius r; G is the gravitational constant.

In the undisturbed region the equilibrium equations are assumed to hold, to which equations (1)–(3) reduce when $v = 0$:

$$\frac{dM}{dr} = 4\pi r^2 \rho, \qquad \frac{1}{\rho}\frac{dp}{dr} + \frac{GM}{r^2} = 0. \tag{5}$$

Equation (4) is replaced by the adiabaticity condition, which is identically satisfied in the region of equilibrium. Condition (4) means that in the disturbed region of the flow there is a high temperature, depending, as a result of intensive heat exchange, only on time (^{3}).

At the front of the shock wave, which begins to propagate at time $t = 0$ through the medium at rest, the following conditions must be satisfied (^{4}):

$$M_1 = M_2, \qquad -\rho_1 c = \rho_2(v_2 - c), \qquad p_1 + \rho_1 c^2 = p_2 + \rho_2(v_2 - c)^2. \tag{6}$$

Here c is the speed of the shock wave. Quantities ahead of the front are furnished with the subscript 1, and those behind the shock-wave front with the subscript 2.

In a dynamic explosion of equilibrium there is no release of energy in the ideal medium at rest (^{1}); consequently, the energy of the moving medium behind the shock-wave front

$$\int_0^{r_2} \left(\frac{\rho v^2}{2} + \rho\varepsilon(p, \rho) - \frac{\rho GM}{r} \right) 4\pi r^2 \, dr \tag{7}$$

must be equal to the initial energy of the medium in the region inside the sphere of radius r_2

$$\int_0^{r_2} \left(\rho_1 \varepsilon(\rho_1, p_1) - \frac{\rho_1 GM}{r} \right) 4\pi r^2 \, dr. \tag{8}$$

Here $\varepsilon(\rho, p)$ denotes the internal energy of an ideal two-parameter medium.

Let, in the equilibrium region, the density of the medium be distributed according to the power law

$$\rho_1(r) = Ar^{-\omega} \quad (1 < \omega < 2.5). \tag{9}$$

Considering media with internal energy of the form

$$\varepsilon(\rho, p) = \frac{p}{\rho} \left\{ \frac{1}{\gamma - 1} + B_1 R^{n_1} + B_2 R^{n_2} + C_1 P^m + C_2 P^{-1} + \int_{q_0}^{q_k} D(q) R^q P^{s(q)} \, dq + \right.$$

Fig. 1

Figure 1: Fig. 1

$$+\int_{h_0}^{h_k}\left[B(h)R^h + C(h)P^{-\frac{\omega}{2(1-\omega)}h} + \sum_{i=1}^{l} D_i(h)R^{q_i}P^{-\frac{\omega}{2(1-\omega)}(h-q_i)}\right]dh\Bigg\} \tag{10}$$

$$\left(R = \frac{\rho}{\rho_0},\ P = \frac{p}{p_0}\right),$$

where ρ_0 and p_0 are certain constants with the dimensions of density and pressure, respectively, we shall find the conditions that must be imposed on the constants and functions entering expression (10) in order that equations (1)–(6) be satisfied and that the quantities (7) and (8) be equal to one another, i.e., that a dynamic explosion of equilibrium take place.

It turns out that the desired solution has the following form: in the equilibrium region, taking (9) into account, the mass and pressure are determined by the formulas

$$M_1(r) = \frac{4\pi A}{3-\omega}\,r^{3-\omega},$$

$$p_1(r) = \frac{2\pi A^2 G}{(3-\omega)(\omega-1)}\,r^{2-2\omega}, \tag{11}$$

in the perturbed region there is an exact solution of equations (1)–(3), taking into account conditions (6) at the shock wave:

$$v = \frac{2}{3}\frac{r}{t}, \qquad \rho = \frac{1}{6\pi G}\frac{1}{t^2}, \qquad M = \frac{2}{9G}\frac{r^3}{t^2},$$

$$p = \frac{(3-\omega)(13\omega-12)}{162\omega(\omega-1)\pi G}\left(\frac{18\pi AG}{3-\omega}\right)^{2/\omega} t^{4(1-\omega)/\omega}, \tag{12}$$

and for the radius of the shock-wave front one obtains the dependence

$$r_2 = \left(\frac{18\pi AG}{3-\omega}\right)^{1/\omega} t^{2/\omega}. \tag{13}$$

Fig. 1

Equating expressions (7) and (8), using formulas (9)–(13), we obtain the equation for determining the constant γ

$$\gamma = \frac{2(8\omega - 15)}{3\omega}, \tag{14}$$

transcendental equations for determining the constants n_1, n_2, m and the function $s(q)$ for a given q:

$$5 - (2+n)\omega = \frac{3\omega}{13\omega - 12}\left(\frac{3-\omega}{3}\right)^{n}; \tag{15}$$

$$5 - 2\omega + 2(1-\omega)m = \frac{3\omega}{13\omega - 12}\left(\frac{3\omega}{13\omega - 12}\right)^{m}; \tag{16}$$

$$5 - 2\omega - \omega q + 2(1-\omega)s(q) = \frac{3\omega}{13\omega - 12}\left(\frac{3-\omega}{3}\right)^{q}\left(\frac{\omega}{13\omega - 12}\right)^{s(q)} \tag{17}$$

and the condition imposed on the functions $B(h)$, $C(h)$, and $D_i(h)$ $(i = 1, 2, \ldots, l)$:

$$\alpha(\omega, h)B(h) + \beta(\omega, h)C(h) + \sum_{i=1}^{l}\delta_i(\omega, h)D_i(h) = 0, \tag{18}$$

where the following notation has been introduced:

$$\alpha(\omega, h) = \frac{8\pi^2 A^2 G}{(3-\omega)(\omega-1)}\left(\frac{A}{\rho_0}\right)^{h}\left\{\frac{13\omega - 12}{3\omega}\left(\frac{3}{3-\omega}\right)^{h} - \frac{1}{5 - \omega(2+h)}\right\},$$

$$\beta(\omega, h) = \frac{8\pi^2 A^2 G}{(3-\omega)(\omega-1)}\left[\frac{2\pi A^2 G}{(3-\omega)(\omega-1)\rho_0}\right]^{-\frac{\omega}{2(1-\omega)}h} \times$$

$$\times\left\{\frac{13\omega - 12}{3\omega}\left(\frac{13\omega - 12}{\omega}\right)^{-\frac{\omega}{2(1-\omega)}h} - \frac{1}{5 - \omega(2+h)}\right\},$$

$$\delta_i(\omega, h) = \frac{8\pi^2 A^2 G}{(3-\omega)(\omega-1)}\left(\frac{A}{\rho_0}\right)^{q_i}\left[\frac{2\pi A^2 G}{(3-\omega)(\omega-1)\rho_0}\right]^{\frac{\omega}{2(\omega-1)}(n-q_i)} \times$$

$$\times\left\{\frac{13\omega - 12}{3\omega}\left(\frac{3}{3-\omega}\right)^{q_i}\left(\frac{13\omega - 12}{\omega}\right)^{\frac{\omega}{2(\omega-1)}(n-q_i)} - \frac{1}{5 - \omega(2+h)}\right\}. \tag{19}$$

The quantity q_0 in formula (10) must satisfy the inequality $q_0 \geq q_*(\omega)$, where for $q = q_*$ the straight line

$$y = \frac{3 - \omega q}{2(1-\omega)} + x \tag{20}$$

is tangent to the exponential curve

$$z = \frac{3}{2(1-\omega)} \left(\frac{3-\omega}{3}\right)^q \left(\frac{\omega}{13\omega - 12}\right)^x . \tag{21}$$

If $1 < \omega < \omega_*$, where $\omega_* = 1.9846$ is the root of the equation

$$\ln \frac{\omega_*}{13\omega_* - 12} = \frac{2(1-\omega_*)}{3 - \omega_*}, \tag{22}$$

then q_k must satisfy the inequality $q_k \leq q_{**}(\omega)$, where for $q = q_{**}$ the straight line (20) and the curve (21) are tangent to one another. If $\omega_* < \omega < 2.5$, the value q_k is not bounded above. h_0, h_k, q_1, $q_2, \dots, q_l$ are certain constants, with $h_k < 5/\omega - 2$.

The constants B_1, B_2, C_1, C_2 and the functions $D(h)$, $B(h)$, $C(h)$, and $D_i(h)$ ($i = 1, 2, \dots, l$) are arbitrary. For the flow to be physically real, the internal energy, defined by formula (10), must be a nonnegative quantity, which imposes a certain additional constraint on these constants and functions. Thus, if $B(h) \equiv C(h) \equiv D_i(h) \equiv 0$,

where $30/13 < \omega < 2.5$, so that, by virtue of (14), $\gamma > 1$; for nonnegativity of the internal energy it is sufficient that the constants B_j, C_j ($j = 1, 2$) and the function $D(h)$ be positive. If $D(h) \equiv B(h) \equiv C(h) \equiv D_i(h) \equiv 0$, and $1 < \omega < 30/13$, so that $\gamma < 1$, the positive constants B_1, B_2, C_1, and C_2 must also satisfy the inequality

$$\frac{1}{\gamma - 1} + (n_1 - n_2) n_1^{\frac{n_1}{n_2 - n_1}} (-n_2)^{-\frac{n_2}{n_2 - n_1}} B_1^{\frac{n_2}{n_2 - n_1}} B_2^{-\frac{n_1}{n_2 - n_1}} + (1+m) m^{-\frac{m}{m+1}} \frac{1}{C_1^{m+1}} C_2^{\frac{m}{m+1}} \geq 0. \tag{23}$$

The temperature T is determined by integrating the linear partial differential equation (4)

$$T + \frac{\partial T}{\partial p}\left(\rho^2 \frac{\partial \varepsilon}{\partial \rho} - p\right) - \rho^2 \frac{\partial T}{\partial p} \frac{\partial \varepsilon}{\partial p} = 0, \tag{24}$$

where $\varepsilon(p, \rho)$ in formula (24) is determined by expression (10).

Fig. 2

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

Fig. 3

The graphs of the functions $n(\omega)$ and $m(\omega)$, determined implicitly by equations (15) and (16), are given in Figs. 1 and 2, respectively. As an example of a solution of equation (17), Fig. 3 gives the graph of the function $s(q,\omega)$ for $q = 1/2$.

If in formula (10) we put

$$B_1 \equiv B_2 \equiv C_1 \equiv C_2 \equiv D(q) \equiv B(h) \equiv C(h) \equiv D_i(h) \equiv 0 \qquad (i = 1, 2, \dots, l),$$

we obtain the solution, found in article [2], of the problem of the dynamic explosion of equilibrium in an ideal gas with zero temperature gradient.

Received
30 VI 1962

REFERENCES

[1] L. I. Sedov, DAN, **112**, No. 2 (1957).
[2] E. V. Ryazanov, DAN, **126**, No. 5 (1959).
[3] V. P. Korobeinikov, DAN, **109**, No. 2 (1956).
[4] L. I. Sedov, *Methods of Similarity and Dimension in Mechanics*, Moscow, 1954.

Note: Figure translations are in progress. See original paper for figures.