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Abstract

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ON THE HOMOGENEOUS PROBLEM OF THE OBLIQUE DERIVATIVE FOR HARMONIC FUNCTIONS IN THREE-DIMENSIONAL DOMAINS

Let D be a bounded simply connected domain of Euclidean space in the variables x, y, z, bounded by a Lyapunov surface S, and let $P = (p, q, r)$ be a prescribed continuous vector field on S.

The oblique derivative problem consists in determining a function $U(x, y, z)$, harmonic in the domain D, continuous together with its first derivatives up to the boundary S, and satisfying the condition

$$(\operatorname{grad} U \cdot P)_S = f, \tag{1}$$

where f is a prescribed continuous function on S.

The plane (two-dimensional) analogue of this problem has been well studied in the case when, at each point of the boundary S of the domain D, the vector $P \neq 0$.

In the notation $P = p + iq,\ \Phi(z) = U_x - iU_y$, condition (1) in the plane case can be written in the form

$$\operatorname{Re}[P(t)\Phi(t)]_S = f(t), \tag{2}$$

and thus the oblique derivative problem turns out to be reduced to the well-studied Hilbert problem: to find a function $\Phi(z)$, holomorphic in the domain D, continuous up to the contour and satisfying the boundary condition (2). Without loss of generality one may assume that D is the unit disk with center at the origin. For the disk this problem is investigated very simply.

In the theory of the plane oblique derivative problem an important role is played by the integer $\varkappa$—the Kronecker index of the vector field P, which, for the adopted positive direction of traversal of the boundary S of the domain D, is expressed in the form

$$\varkappa = \frac{1}{2\pi} \int_0^{2\pi} (p^2 + q^2)^{-1} \begin{vmatrix} p & q \\ p_\varphi & q_\varphi \end{vmatrix} d\varphi = \frac{1}{2\pi} \operatorname{Var}[\arg(p + iq)]_S$$

and characterizes the rotation of the vector field P, or the degree of the mapping.

The following classical results are known:

The plane homogeneous oblique derivative problem for $\varkappa \le 0$ has exactly $-2\varkappa+2$ linearly independent solutions, while for $\varkappa \ge 1$ its solution is only a constant (arbitrary). The nonhomogeneous plane problem for $\varkappa \le 0$ is always solvable and its general solution contains linearly $-2\varkappa+2$ arbitrary constants, while for $\varkappa \ge 1$ this problem is solvable if and only if the given function f is subject to $2\varkappa - 1$ integral orthogonality conditions along the contour S.

The plane oblique derivative problem has also been investigated by the method of one-dimensional integral equations with special kernels of Cauchy type, whose theory is well developed thanks to the existence, again, of the theory of the Hilbert problem (2).

The spatial oblique derivative problem has been investigated more or less well only in the case when the vector P at no point lies in the tangent plane of the surface S. In the case, however, when many-

the set E of points of tangency of the vector field P with the surface S is nonempty, considerable difficulties arise in the study of the indicated problem.

It is true that, adopting the notation $\Phi = \operatorname{grad} U$, this problem can be interpreted as a three-dimensional analogue of Hilbert' s problem for holomorphic vectors $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ that are regular solutions of the system

$$\operatorname{div}\Phi = 0, \qquad \operatorname{rot}\Phi = 0$$

and satisfy the boundary condition

$$(\Phi \cdot P)_S = f,$$

but such an interpretation, owing to the absence of a suitable theory of multidimensional singular integral equations with Cauchy kernels, is as yet of little use.

Moreover, the Kronecker index

$$\varkappa = \frac{1}{4\pi}\iint_S \begin{vmatrix} p & q & r \\ p_\varphi & q_\varphi & r_\varphi \\ p_\vartheta & q_\vartheta & r_\vartheta \end{vmatrix} \frac{d\varphi\, d\vartheta}{(p^2+q^2+r^2)^{3/2}},$$

which characterizes the rotation of the three-dimensional vector field P, ceases to play exactly the same role as it plays in the theory of the planar oblique-derivative problem.

In particular, this is easily seen from the simple example of the homogeneous spatial oblique-derivative problem

$$(U_z)_S = 0,$$

where S is a sphere. Indeed, in this case, as is easy to see, $\varkappa = 0$. However, a solution of the problem is an arbitrary harmonic function of two variables $F(x, y)$.

In the present note an attempt is made to establish some simple facts concerning the spatial homogeneous oblique-derivative problem for harmonic functions $U(x, y, z) \equiv U(Q)$

$$(\operatorname{grad} U \cdot P)_S = 0 \tag{3}$$

in the case where the above-mentioned set E is nonempty.

In what follows, alongside the well-known extremum principle, we shall need one more property of functions harmonic in the domain D, first established by Zaremba.

Lemma (Zaremba). *If a harmonic function U, distinct from a constant, in the domain D, continuous together with its partial derivatives up to the boundary S, attains a maximum (minimum) at some point $M \in S$, then at this point*

$$\frac{\partial U}{\partial l} > 0 \qquad \left(\frac{\partial U}{\partial l} < 0\right),$$

where l is a vector with origin at the point M, directed away from D and satisfying the condition $\cos \widehat{Nl} > 0$, while N is the outward normal to the boundary S at the same point.

Theorem 1. *If the set E consists of a single point M, then problem (3) cannot have a solution distinct from a constant.*

Suppose that problem (3) has a solution $U(Q)$ distinct from a constant, and consider the function $V(Q) = U(Q) - U(M)$.

In view of the fact that $V(Q)$ is a solution of problem (3), on the basis of the lemma we conclude that this function cannot attain either a maximum or a minimum at points of the set $S - \{M\}$. Hence, since $V(M) = 0$, we conclude that $V \equiv 0$ everywhere in the domain D, i.e. $U(Q) \equiv U(M)$.

Theorem 2. *If the set E consists of n points $M_1, M_2, \dots, M_n$, then the number of linearly independent solutions of problem (3) cannot exceed n.*

Assume the contrary and denote by $U_1, U_2, \dots, U_n$ linearly independent solutions of problem (3). Let $U_{n+1}(Q)$ be a solution of problem (3) linearly independent of $U_1, U_2, \dots, U_n$. The equalities $U_{n+1}(M_k) = 0$, $k = 1, 2, \dots, n$, cannot hold, since, in the presence of these equalities, by condition (3) and the lemma we would conclude that $U_{n+1}(Q) \equiv 0$.

By virtue of the linear independence of the functions $U_1, U_2, \dots, U_n$, the determinant $\det U_k(M_j) \neq 0$. Otherwise the system

$$\sum_{k=1}^{n} C_k U_k(M_j) = 0, \qquad j = 1, 2, \dots, n,$$

would have a nontrivial solution $C_1^0, C_2^0, \dots, C_n^0$, and the function

$$U(Q) = \sum_{k=1}^{n} C_k^0 U_k(Q)$$

would be a solution of problem (3) vanishing at the points M_j, $j = 1, 2, \dots, n$. Hence, by the lemma, we would conclude that $U(Q) \equiv 0$, which contradicts the linear independence of the functions $U_1, U_2, \dots, U_n$. Thus the system

$$\sum_{k=1}^{n} C_k U_k(M_j) = U_{n+1}(M_j), \qquad j = 1, 2, \dots, n, \tag{4}$$

has a solution $C_1, C_2, \dots, C_n$.

Now consider the function $V = \sum_{k=1}^{n} C_k U_k(Q) - U_{n+1}(Q)$. By virtue of equalities (3) and (4), on the basis of the lemma we conclude that $V \equiv 0$, and therefore $U_{n+1}(Q) \equiv \sum C_k U_k(Q)$.

Theorem 3. *If the set E is a smooth arc Γ, at each point of which the direction P coincides with the direction of the tangent to this curve, then problem (3) cannot have a solution different from a constant.*

Let $U(Q)$ be a solution of problem (3). By virtue of (3) and the condition of the theorem, $(U)_\Gamma = \text{const}$. The function $V(Q) = U(Q) - \text{const}$ is a solution of problem (3) vanishing on Γ. Therefore, by the lemma, $U(Q) - \text{const} = 0$ everywhere in the domain D.

Theorem 4. *If the set E consists of n pairwise nonintersecting smooth arcs, at each point of which the direction of the tangent to the arc coincides with the direction P, then the number of linearly independent solutions of problem (3) cannot exceed n.*

Theorem 4 is proved by repeating the arguments used in the proofs of Theorems 2 and 3.

It is easy to establish a sufficient condition for a solution U of problem (3), different from a constant, not to attain either a maximum or a minimum at a point $M \in E$.

Indeed, suppose that the function U attains a maximum at the point M. By the lemma, at the point M the inequality

$$\partial U/\partial N > 0 \tag{5}$$

must hold.

In a neighborhood of the point M, introduce on the surface S the internal coordinates φ, ϑ. At the point M, along with inequality (5), the equalities $U_\varphi = U_\vartheta = 0$ hold.

On the surface S one can indicate a sequence of points converging to the point M in the direction of increasing φ and ϑ, such that along this sequence the equalities

$$U_x x_\varphi + U_y y_\varphi + U_z z_\varphi = \lambda^2, \qquad U_x x_\vartheta + U_y y_\vartheta + U_z z_\vartheta = \mu^2, \tag{6}$$

hold, where λ^2 and μ^2 do not vanish simultaneously. If $\lambda^2 = \mu^2 = 0$

along the chosen sequence of points, then, by virtue of the linear dependence of the vectors $P(p, q, r)$, $S_\varphi(x_\infty, y_\infty, z_\infty)$, $S_\theta(x_\theta, y_\theta, z_\theta)$, we would obtain that at the point M the equalities $U_x = U_y = U_z = 0$ hold, which contradicts condition (5).

From (3) and (6) we have

$$\frac{\partial U}{\partial N} = \frac{1}{[P, S_\varphi, S_\theta]}\{\lambda^2[P, N, S_\theta] + \mu^2[P, N, S_\varphi]\}, \tag{7}$$

where N denotes the outward normal to the surface S, and the square brackets denote the mixed product of three vectors.

After this we are immediately convinced of the validity of the following theorem:

Theorem 5. If, along the chosen sequence of points, the sign of the right-hand side of formula (7) is negative, then at the point M we shall have $\partial U/\partial N < 0$, which excludes inequality (5), and, consequently, at the point M the function U cannot attain a maximum. If only the condition just formulated is satisfied, at the point M the function U cannot attain a minimum either.

From Theorem 5, in particular, it follows that, if for m points ($m \leq n$; it is assumed that E consists of n points) the sign of the right-hand side of formula (7) is negative, then the number of linearly independent solutions of problem (3) cannot exceed $n - m$.

The validity of this assertion is revealed by reasoning analogous to the reasoning given above in the proof of Theorem 3.

In the case of the planar homogeneous problem of the oblique derivative, analogous reasoning gives, instead of (7), the expression

$$\frac{\partial U}{\partial N} = -\lambda^2 \frac{(P \cdot S_\varphi)}{(P \cdot N)},$$

where N is the outward normal, and S_φ is the tangent to the boundary S of the domain D. The condition that, as one approaches the point M, the expressions $(P \cdot S_\varphi)$ and $(P \cdot N)$ have the same signs excludes the possibility that the function $U(x, y)$ attain an extremum at the point M. This condition means that, as one approaches the point M, the rotation of the field P occurs in the positive direction; this is fully consistent with the classical results cited above concerning the planar problem.

The fact established by Theorem 5 may prove useful in finding an integral condition for the absence of solutions other than the constant ones of the homogeneous spatial problem of the oblique derivative, and also a method for exact computation of the number of linearly independent solutions when they exist.

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Note: Figure translations are in progress. See original paper for figures.