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Abstract

Full Text

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ON THE SPECTRUM OF SINGULAR NON-SELF-ADJOINT DIFFERENTIAL OPERATORS OF ORDER $2n$

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Let us consider the differential expression

$$l(y) \equiv (-1)^n \left(P_0(x) y^{(n)}\right)^{(n)} + (-1)^{n-1} \left(P_1(x) y^{(n-1)}\right)^{(n-1)} + \dots + P_n(x) y, \qquad 0 \le x < \infty, \tag{1}$$

where $1/P_0(x),\ P_1(x), \dots, P_n(x)$ are complex-valued functions locally summable on the interval $R^+ = [0, \infty)$.

Denote by D the totality of all functions $y(x) \in L^2(R^+)$ such that:
1) the quasi-derivatives $y^{[i]}(x)$, $i = 0, 1, \dots, 2n-1$, exist and are absolutely continuous on every finite interval $[0, a]$, $a > 0$; 2) $l(y) \in L^2(R^+)$.

Suppose that boundary conditions are given,

$$U_i(y) \equiv \sum_{j=0}^{2n-1} \theta_{ij} y^{[j]}(0) = 0, \qquad i = 1, 2, \dots, n, \tag{2}$$

where θ_{ij} are complex numbers. Denote by D_θ the totality of all functions $y(x) \in D$ satisfying conditions (2). In the space $L^2(R^+)$ define the operator L_θ as follows: its domain of definition is D_θ, and for $y(x) \in D_\theta$, $L_\theta y = l(y)$.

The non-self-adjoint operator L_θ in the case $n = 1$, $P_0(x) \equiv 1$, for complex $P_n(x)$ and complex θ, was first considered by M. A. Naimark (^{1a}), and since then has been the subject of investigations by many authors. In the present note we set forth some results on the spectrum of the operator L_θ under various assumptions concerning $P_0(x),\ P_1(x), \dots, P_n(x)$.

1°. Assume that the functions $\left(\operatorname{Re} \frac{1}{P_0(x)}\right)'$, $\operatorname{Im} \frac{1}{P_0(x)}$, $P_1(x), \dots, P_n(x) \in L(R^+)$, and let

$$\lim_{x \to \infty} \operatorname{Re} \frac{1}{P_0(x)} > 0.$$

It can be shown that under these conditions the differential equation $l(y) = \lambda y$ has $2n$ linearly independent solutions $y_1(x), y_2(x), \dots, y_{2n}(x)$ such that, as $x \to +\infty$,

$$y_j^{[i]}(x) = c_i(x)\rho_j^i e^{\rho_j \xi(x)}[1 + o(1)], \qquad i = 0, 1, \dots, 2n-1;\ j = 1, 2, \dots, 2n, \quad (3)$$

where

$$\xi(x) = \int_0^x \sqrt[2n]{\operatorname{Re}\frac{1}{P_0(t)}}\, dt;$$

$$c_i(x) = \begin{cases} 1, & \text{for } i \le n-1, \\ (-1)^{i-n}\left(\operatorname{Re}\dfrac{1}{P_0(x)}\right)^{-1}, & \text{for } i \ge n, \end{cases}$$

and where $\rho_1, \rho_2, \dots, \rho_{2n}$ are the distinct roots of degree $2n$ of $(-1)^n\lambda$.

We divide the complex s-plane into $2n$ equal sectors S_k, defined by the inequality

$$\frac{k\pi}{n} < \arg s < \frac{k+1}{n}\pi.$$

Let $\omega_1, \omega_2, \dots, \omega_{2n}$ be all the distinct roots of the $2n$-th degree of unity. In each sector S_k one can choose an arrangement of the numbers $\omega_1, \omega_2, \dots, \omega_{2n}$ such that, for $s \in S_k$,

$$\operatorname{Re} s\omega_1 < \operatorname{Re} s\omega_2 < \dots < \operatorname{Re} s\omega_{2n}.$$

It follows from formulas (3) that, for $s \in S_k$, among the functions $y_1(x), y_2(x), \dots, y_{2n}(x)$ exactly n functions belong to $L^2(R^+)$, while the remaining n functions, and no nonzero linear combination of them, will belong to $L^2(R^+)$. Let $y_1(x), y_2(x), \dots, y_n(x)$ be those solutions of the equation $l(y) = \lambda y$ which belong to $L^2(R^+)$. It is clear that the eigenfunctions of the operator L_θ can only be linear combinations of these functions: $y(x) = c_1y_1(x) + c_2y_2(x) + \dots + c_ny_n(x)$. Since $y(x) \in D_\theta$, the corresponding eigenvalues of the operator L_θ are determined from the equation

$$A(s) = \begin{vmatrix} U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) \\ U_2(y_1) & U_2(y_2) & \cdots & U_2(y_n) \\ \cdots & \cdots & \cdots & \cdots \\ U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) \end{vmatrix} = 0.$$

Let the complex number $\lambda = (-1)^n s^{2n}$, $s \in S_k$, not be an eigenvalue of the operator L_θ; taking into account that $y_1(x), y_2(x), \ldots, y_n(x) \in L^2(R^+)$, while $y_{n+1}(x), y_{n+2}(x), \ldots, y_{2n}(x) \notin L^2(R^+)$, and using the boundary conditions (2), we compute the resolvent of the operator L_θ:

$$R_\lambda f = \int_0^\infty K(x,t,\lambda) f(t)\,dt,$$

where $f(x) \in L^2(R^+)$,

$$K(x,t,\lambda) = \begin{cases} \displaystyle\sum_{k=1}^n y_k(t,s)\,[h_k(x,s) - v_k(x,s)], & \text{for } x < t, \\ \displaystyle\sum_{k=1}^n y_k(x,s)\,[h_k(t,s) - v_k(t,s)], & \text{for } x > t; \end{cases}$$

moreover $h_k(x,s) \in L^2(R^+)$, $k = 1, 2, \ldots, n$, and the functions $v_k(x,s)$ have the form

$$v_k(x,s) = a_k e^{-s\omega_k \xi(x)}[1 + o(1)], \qquad k = 1, 2, \ldots, n,$$

where a_k are certain constants.

Theorem 1. Suppose the functions $\left(\operatorname{Re}\frac{1}{P_0(x)}\right)'$, $\operatorname{Im}\frac{1}{P_0(x)}$, $P_1(x), \ldots, P_n(x)$ belong to $L(R^+)$, and suppose

$$\lim_{x\to\infty} \operatorname{Re}\frac{1}{P_0(x)} > 0.$$

Then the spectrum of the operator L_θ is continuous on the positive half-axis and discrete in the entire remaining complex λ-plane; moreover, the limit points of the nonreal spectrum and, for $n > 1$, the eigenvalues of the operator L_θ may lie on the positive half-axis. For the remaining values of λ not belonging to the spectrum, the resolvent R_λ of the operator L_θ is an integral operator with kernel $K(x,t,\lambda)$ satisfying the conditions

$$\int_0^\infty |K(x,t,\lambda)|^2\,dx < \infty, \qquad \int_0^\infty |K(x,t,\lambda)|^2\,dt < \infty. \tag{4}$$

Suppose $\operatorname{Re}\frac{1}{P_0(x)}$ is not differentiable, but there exists a number $a_0 > 0$ such that

$$\operatorname{Re}\frac{1}{P_0(x)} - a_0 \in L(R^+),$$

and suppose

$$\operatorname{Im}\frac{1}{P_0(x)},\ P_1(x),\dots,P_n(x)\in L(R^+).$$

then the equation $l(y)=\lambda y$ has $2n$ linearly independent solutions such that, as $x\to+\infty$,

$$y_j^{[i]}(x)=c_i(s\omega_j)^i e^{\rho\omega_j(x)}[1+o(1)],\qquad \rho=s\sqrt[2n]{\frac{1}{a_0}},$$

$$i=0,1,\dots,2n-1;\qquad j=1,2,\dots,2n;$$

where c_i are constants. In this case the assertion of Theorem 1 also remains valid.

2°. Let $\operatorname{Re}P_n(x)\to+\infty$ as $x\to+\infty$, and suppose that the following conditions are satisfied:

a) for sufficiently large x_0, the functions $(\operatorname{Re}P_n(x))'$ and $(\operatorname{Re}P_n(x))''$ do not change sign on the interval $[x_0,\infty)$;

b) as $x\to\infty$,

$$(\operatorname{Re}P_n(x))'=o\left(|\operatorname{Re}P_n(x)|^\alpha\right),\qquad 0<\alpha<1+\frac{1}{2n};$$

c) the functions

$$\left(\operatorname{Re}\frac{1}{P_0(x)}\right)^{-1}\cdot\left(\operatorname{Re}\frac{1}{P_0(x)}\right)',\qquad \left(\operatorname{Re}\frac{1}{P_0(x)}\right)'\operatorname{Im}\frac{1}{P_0(x)}|\operatorname{Re}P_n|^{\frac{1}{2n}},$$

$$P_1(x)|\operatorname{Re}P_n(x)|^{-\frac{1}{2n}},\dots,P_{n-1}(x)|\operatorname{Re}P_n(x)|^{\frac{-2n+3}{2}},\operatorname{Im}P_n(x)|\operatorname{Re}P_n(x)|^{\frac{-2n+1}{2}}$$

are summable on the interval R^+.

Then the equation $l(y)=\lambda y$ has $2n$ linearly independent solutions $y_1(x),y_2(x),\dots,y_{2n}(x)$ such that, as $x\to+\infty$,

$$y_j^{[i]}(x)=c_i(x)\omega_j^i e^{\omega_j\xi(x)}[1+o(1)],\tag{5}$$

where

$$c_i(x) = \begin{cases} \left(\operatorname{Re}\dfrac{1}{P_0(x)}\right)^{1/2} \rho^{\frac{-2n-1}{2}+i}, & \text{for } i \le n-1, \\ (-1)^{i-n}\left(\operatorname{Re}\dfrac{1}{P_0(x)}\right)^{-1/2} \rho^{\frac{-2n-1}{2}+i}, & \text{for } i \ge n; \end{cases}$$

$$\xi(x) = \int_{x_0}^{x} \rho(t)\,dt = \int_{x_0}^{x} \sqrt[2n]{(-1)^n(\lambda - \operatorname{Re} P_n(t))\operatorname{Re}\frac{1}{P_0(t)}}\,dt.$$

Here $\arg\rho(x)$ is determined by continuity from a single-valued choice of its value at a fixed value of x.

Theorem 2. Let $\operatorname{Re} P_n(x) \to \infty$ as $x \to +\infty$, and let conditions a), b), c) be satisfied. Then the spectrum of the operator L_θ is discrete and has no finite limit points. For all other values of λ not belonging to the spectrum, the resolvent R_λ is an integral operator with a kernel satisfying conditions (4).

3°. Suppose now that $\operatorname{Re} P_n(x) \to -\infty$ as $x \to +\infty$. From the asymptotic formulas (5) it follows that the number of linearly independent solutions of the equation $l(y) = \lambda y$ belonging to $L^2(R^+)$ will be $n+1$ or n, depending on whether the integral

$$\int_0^\infty |\operatorname{Re} P_n(t)|^{-1+\frac{1}{2n}}\,dt \tag{6}$$

converges or diverges.

Theorem 3. Let $\operatorname{Re} P_n(x) \to -\infty$ as $x \to +\infty$, and let the integral (6) diverge; moreover, let conditions a), b), and c) be satisfied. Then the spectrum of the operator L_θ fills the entire real axis, while in the remaining part of the complex λ-plane the spectrum can only be discrete. For all other values of λ not belonging to the spectrum, the resolvent R_λ is an integral operator with a kernel satisfying conditions (4).

In the case when the integral (6) converges, instead of the operator L_θ another operator $\hat{L}$ is considered, defined as follows: denote by D^* the totality of functions $z(x) \in L^2(R^+)$, analogous to D, but constructed for the adjoint differential expression

$$l^*(y) = (-1)^n(\overline{P_0}(x)z^{(n)})^{(n)} + (-1)^{n-1}(\overline{P_1}(x)z^{(n-1)})^{(n-1)} + \cdots + \overline{P_n}(x)z.$$

Let $y(x) \in D,\ z(x) \in D^*$; then from Lagrange' s formula

$$\int_\alpha^\beta [\, l(y)\overline{z} - y\overline{l^*(z)}\,]\,dt = [y,z]_\alpha^\beta,$$

where

$$[y, z] = \sum_{k=1}^{2n-1} y^{[2n-k]}\overline{z^{[k-1]}} - y^{[k-1]}\overline{z^{[2n-k]}},$$

it follows that $[y, z]_0^\infty$ exists. Choose functions $z_1(x), z_2(x), \dots, z_{n+1}(x)$ from D^* such that the determinant

$$\Delta(\lambda) = \begin{vmatrix} [y_1, z_1]_0^\infty & [y_2, z_1]_0^\infty & \cdots & [y_{n+1}, z_1]_0^\infty \\ [y_1, z_2]_0^\infty & [y_2, z_2]_0^\infty & \cdots & [y_{n+1}, z_2]_0^\infty \\ \cdots & \cdots & \cdots & \cdots \\ [y_1, z_{n+1}]_0^\infty & [y_2, z_{n+1}]_0^\infty & \cdots & [y_{n+1}, z_{n+1}]_0^\infty \end{vmatrix}$$

does not vanish identically. Denote by $\hat{D}$ the set of all functions $y(x) \in D$ satisfying the conditions $[y, z_1]_0^\infty = 0,\ [y, z_2]_0^\infty = 0, \dots, [y, z_{n+1}]_0^\infty = 0$, and by $\hat{L}$ the operator with domain $\hat{D}$ such that $\hat{L}y = l(y)$ for $y \in \hat{D}$.

Computing the resolvent $\hat{R}_\lambda$ of the operator $\hat{L}$, we arrive at the following result.

Theorem 4. *Let* $\operatorname{Re} P_n(x) \to -\infty$ *as* $x \to +\infty$*, let the integral (6) converge, and let conditions a), b), and c) be fulfilled. Then the spectrum of the operator* $\hat{L}$ *is discrete and has no finite limit points. For values of* λ *not belonging to the spectrum, the resolvent* $\hat{R}_\lambda$ *of the operator* $\hat{L}$ *is an integral operator with a Hilbert-Schmidt kernel.*

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.