MATHEMATICS

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Abstract

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THE PLANCHEREL FORMULA FOR THE GROUP OF UNIMODULAR MATRICES OF THE SECOND ORDER WITH ELEMENTS FROM A LOCALLY COMPACT FIELD

1. We consider the group G of unimodular matrices of the second order with elements from a nondiscrete locally compact field K. In paper (^{1}) a description was given of the irreducible unitary representations of the group G. Namely, it was established that the group G has several series of irreducible unitary representations. One of these series (the "continuous" series) is connected with the basic field K; each of the remaining ("discrete") series is connected with one of the quadratic extensions of the field K. Thus, if K is the field of complex numbers, then there is only one series (since the field of complex numbers has no extensions); if K is the field of real numbers, then there are 2 series of representations (since the field of real numbers has one quadratic extension); if K is a disconnected field, then there are 4 series of representations (since a disconnected field has 3 quadratic extensions*). Within each series the representation is specified by a certain multiplicative character. More precisely, a representation of the continuous series is specified by a multiplicative character π on K; here the characters π and π^{-1} correspond to equivalent representations. A representation of the discrete series, connected with a quadratic extension $K(\sqrt{\tau})$ of the field K, is specified by a character π on the "unit circle" $\bar{t}t \equiv x^2 - \tau y^2 = 1$, $t = x + \sqrt{\tau}\, y$; here the characters π and π^{-1} again correspond to equivalent representations.

In paper (^{1}) the traces of the irreducible unitary representations were also found.

We note that in the case of a disconnected field K there is also one more special representation of the group G, which was not indicated in (^{1}). This representation is realized in the space of functions $\varphi(x)$ on K for which

$$\int \varphi(x)\, dx = 0,$$

$$(\varphi, \varphi) = \int \ln|x_1 - x_2|\, \varphi(x_1)\overline{\varphi(x_2)}\, dx_1\, dx_2' < \infty.$$

The representation operator corresponding to the matrix

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

has the form**

$$T_0(g)\varphi(x) = \varphi\left(\frac{\delta x + \beta}{\gamma x + \alpha}\right)|\gamma x + \alpha|^{-2}.$$

The trace $\operatorname{Tr} T_0(g)$ of the operator of the special representation is expressed by the following formula. If the eigenvalues λ, λ^{-1} of the matrix g belong to the field K, then

$$\operatorname{Tr} T_0(g) = \frac{|\lambda| + |\lambda^{-1}|}{|\lambda - \lambda^{-1}|} - 1;$$

if λ, λ^{-1} do not belong to K, then $\operatorname{Tr} T_0(g) = -1$. Here $|\lambda|$ denotes the norm of the element λ in K.

The purpose of the present paper is to decompose functions on the group G with respect to irreducible representations. The precise formulation of the problem is given in § 2.

2. Let $f(g)$ be a finite function on the group G. To each irreducible unitary representation $T_\pi(g)$ of the group G assign the operator

$$T_\pi(f) = \int f(g) T_\pi(g)\, dg. \tag{1}$$

The problem is to obtain the inversion of formula (1), i.e. to recover the function $f(g)$, knowing the operators $T_\pi(f)$. Below a solution of this problem is given for an **unconnected** locally compact field K*.

We shall assume that the finite residue field O/P associated with K, where O is the ring of integers of K, P is the maximal ideal in O, has characteristic different

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* We shall henceforth exclude the special case when the finite residue field O/P connected with K (here O is

In this special case the field K has more than three quadratic extensions (infinitely many of them, if the char

** In the case of the field of real numbers this representation belongs to the discrete series.

from 2. In this case the field K has three quadratic extensions: $K(\sqrt{p})$, $K(\varepsilon p)$, and $K(\sqrt{\varepsilon})$, where p is a generator of the ideal P, and ε is an element of K of finite order $q-1$ (q is the order of the field O/P).

We introduce the following notation. Let π be a multiplicative character on K; let π_τ be a character on the circle $t\bar{t}=1$ in $K(\sqrt{\tau})$. Let d^*t be the multiplicatively invariant measure on K, $d^*_\tau t$ the invariant measure on the circle $t\bar{t}=1$ in $K(\sqrt{\tau})$; $d\pi$, $d\pi_\tau$ the invariant measures on the corresponding groups of characters. We normalize these measures by the following conditions:

$$\int_{|t|\le 1} |t|\, d^*t = 1, \quad \int d^*_\tau t = 1, \quad \int f(t)\pi(t)\, d^*t\, d\pi = f(1), \quad \int f(t)\pi_\tau(t)\, d^*_\tau t\, d\pi_\tau = f(1).$$

Denote by $T_\pi(g)$ the representation of the continuous series corresponding to the character π; by $T_{\pi_\tau}(g)$ the direct sum of the representations of the discrete series $T^+_{\pi_\tau}(g)$ and $T^-_{\pi_\tau}(g)$, corresponding to the character π_τ (see (1)). Finally, by $T_0(g)$ we denote the operator of the special representation.

The following inversion formula holds:

$$\begin{aligned} cf(g) = \int \mu(\pi)\, \mathrm{Tr}(T_\pi(f)T_\pi^{-1}(g))\, d\pi + \\ + \sum_{\tau=p,\ \varepsilon p,\ \varepsilon} \int \mu(\pi_\tau)\, \mathrm{Tr}(T_{\pi_\tau}(f)T_{\pi_\tau}^{-1}(g))\, d\pi_\tau + 2\,\mathrm{Tr}(T_0(f)T_0^{-1}(g)), \end{aligned} \tag{2}$$

where

$$\mu(\pi) = -\int_K \pi(t)|t|\,|1-t|^{-2}\, d^*t \ \ **, \tag{3}$$

$$\mu(\pi_\varepsilon) = -\int_{t\bar{t}=1} \pi_\varepsilon(t)|1-t|^{-2}\, d^*_\varepsilon t, \tag{4}$$

$$\mu(\pi_\tau) = -\int_{t\bar{t}=1,\ |1-t|<1} \pi_\tau(t)\big[|1-t|^{-2}+1\big]\, d^*_\tau t \quad (\tau = p,\ \varepsilon p); \tag{5}$$

$$c = q^{-1}(q-1)^{-1}(q+1);$$

$\mathrm{Tr}\, A$ is the trace of the operator A.

Let us note that the integrals (3), (4), (5) diverge, and therefore they should be understood in the sense of the regularized value. For example, $\mu(\pi_\varepsilon)$ is the value of the analytic function of ν,

$$\varphi(\nu) = \int \pi_\varepsilon(t)|1-t|^\nu \, d_\varepsilon^* t$$

at $\nu = -2$.

* For connected fields the solution was obtained earlier: in $(^2)$ for the field of complex numbers and in $(^3)$ for the field of real numbers.

** For $\pi(x) = |x|^{i\rho}$ the function $\mu(\pi)$ was defined for fields of characteristic 0 in $(^4)$.

From formula (2) the **Plancherel formula** easily follows

$$c \int |f(g)|^2 \, dg = \int \mu(\pi) \operatorname{Tr}(T_\pi(f) T_\pi^*(f)) \, d\pi +$$

$$+ \sum_{\tau = p, \varepsilon p, \varepsilon} \int \mu(\pi_\tau) \operatorname{Tr}(T_{\pi_\tau}(f) T_{\pi_\tau}^*(f)) \, d\pi_\tau + 2 \operatorname{Tr}(T_0(f) T_0^*(f)), \tag{6}$$

where $\mu(\pi)$, $\mu(\pi_\tau)$ are expressed by formulas (3)–(5). This formula is valid for any square-integrable function $f(g)$ on the group G.

3. For connected fields the measure $\mu(\pi)$ entering the Plancherel formula was computed in $(^2,^3)$. In the case of the field of complex numbers it has the form $\mu(\pi) = c(\rho^2 + n^2)$, where $\pi(z) = |z|^{i\rho} e^{in \arg z}$. In the case of the field of real numbers it has the following form. For representations of the continuous series $\mu(\pi) = c\rho \operatorname{cth} \frac{\pi\rho}{2}$, when $\pi(x) = |x|^{i\rho}$; $\mu(\pi) = c\rho \operatorname{th} \frac{\pi\rho}{2}$, when $\pi(x) = |x|^{i\rho} \operatorname{sign} x$. For representations of the discrete series $\mu(\pi) = c|n|$, where $\pi(z) = e^{in \arg z}$.

It is easy to show that for the measure $\mu(\pi)$, both in the case of the field of complex numbers and in the case of the field of real numbers, there is a single formula

$$\mu(\pi) = c \int \pi(t)|t| \, |1-t|^{-2} \, d^* t. \tag{7}$$

In the case of representations of the continuous series, the integral (7) is taken over the principal field K, and in the case of representations of the discrete series—over the circle $\bar{t}t = 1$; d^*t is the measure invariant with respect to multiplication*.

Thus the Plancherel measure in the case of a connected field is expressed by the same formula as in the case of a nonconnected field.

4. The inversion formula (2) can be obtained by direct computation. First of all, note that it is equivalent to the following formula for generalized functions:

$$\mathcal{J}(g) = \int \mu(\pi)\,\mathrm{Tr}\,T_\pi(g)\,d\pi + \sum_{\tau=p,\varepsilon p,\varepsilon} \int \mu(\pi_\tau)\,\mathrm{Tr}\,T_{\pi_\tau}(g)\,d\pi_\tau + 2\,\mathrm{Tr}\,T_0(g) = c\delta(g), \tag{8}$$

where $\delta(g)$ is the delta-function on the group G.

Substitute in the left-hand side of the equality the expressions for $\mu(\pi)$ and $\mu(\pi_\tau)$ and the expressions for the traces (see (^{1})). For $g \neq e$, where e is the identity of the group, all the integrals in (8) can be computed directly. It then turns out that $\mathcal{J}(g) = 0$, when $g \neq e$. Hence it follows that the generalized function $\mathcal{J}(g)$ is concentrated at the point $g = e$, and therefore $\mathcal{J}(g) = c\delta(g)$.

The same reasoning is also valid for the case of a connected field K.

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* For any field K we define the function $|x|$ by the formula $d(xx_0) = |x_0|\,dx$, where dx is the measure on K invariant with respect to addition. Therefore, in the case of the field of complex numbers $|x|$ is the **square** of the modulus of the complex number x.

Note: Figure translations are in progress. See original paper for figures.

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