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Abstract

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On a Nonlinear Theory of the Onset of Turbulence

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I. To describe the initial stage of the onset of turbulence in flows of a viscous incompressible fluid, in ([1]) a closed system of equations was introduced for the mean flow velocity $\mathbf{U}(\mathbf{x}, t)$ and the spectral tensor of turbulent fluctuations

$$\Phi_{ij}(\mathbf{k}, \mathbf{x}, \mathbf{r}) = \int_{-\infty}^{\infty} e^{i\mathbf{kt}}\, \overline{\delta V_i(\mathbf{x} - \mathbf{r}/2)\, \delta V_j(\mathbf{x} + \mathbf{r}/2)}\, d\mathbf{r}.$$

The linear approximation for $\delta\mathbf{V}(\mathbf{x}, t)$ (in the Navier–Stokes equations) corresponds to a closed system of equations obtained by discarding third moments, which are small at the beginning. At a low level of turbulence the fourth moments are expressed through the second moments according to the Gaussian law. However, as the amplitude of the fluctuations increases, their distribution law may differ more and more from the Gaussian one.

In the present work, in order to construct a closed system of equations describing the onset and development of turbulence at fluctuation amplitudes larger than in the initial stage considered in ([1]), the following hypothesis is introduced concerning the relation between higher moments of turbulent fluctuations:

$$b_{j_1 j_2 j_3 j_4 j_5} = \sum_{\substack{k>l;\, p,m,q \neq k,l; \\ l=2,\dots,5}} b_{i_k j_l}\, b_{j_p j_m j_q}. \tag{1}$$

Here $b_{j_1 j_2 j_3 j_4 j_5}$ is the fifth central moment,

$$b_{j_1 j_2 j_3 j_4 j_5} = \overline{\delta V_{j_1}(\mathbf{x}_1, t) \dots \delta V_{j_5}(\mathbf{x}_5, t)}; \qquad b_{j_k j_l} = \overline{\delta V_{j_k}(\mathbf{x}_k, t) \delta V_{j_l}(\mathbf{x}_l, t)},$$

the sum being taken over all partitions of the group of indices $j_1 - j_5$ into two groups of three and two elements.

If, for the multidimensional probability density $P(\delta V_{i_1}, \delta V_{i_2}, \dots, \delta V_{i_n})$, there exists the integral

$$\int_{-\infty}^{\infty} P^2(\delta V_{i_1}, \dots, \delta V_{i_n})\, d\delta V_{i_1} \dots d\delta V_{i_n} = D < \infty, \tag{2}$$

then its expansion in a series with respect to the Gaussian law converges in the mean-square sense, by virtue of the completeness in L^2 of the system of Chebyshev–Hermite orthogonal functions:

$$\dot{P}(\delta V_{i_1}, \dots, \delta V_{i_n}) = P_0 + b_{i_k i_l i_m} \frac{\partial^3}{\partial \delta V_{i_k}\, \partial \delta V_{i_l}\, \partial \delta V_{i_m}} P_0 + \dots \tag{3}$$

Here

$$P_0(\delta V_{i_1}, \dots, \delta V_{i_n}) = \frac{1}{(2\pi B)^{n/2}} \exp\left\{ -\sum_{k,l=1}^{n} \delta V_{i_k} \delta V_{i_l} D_{kl} \right\};$$

$$\|D_{kl}\| = \|b_{i_k i_l}\|^{-1}; \qquad B = \|b_{i_k i_l}\|; \qquad b_{i_k i_l i_m} = \overline{\delta V_{i_k} \delta V_{i_l} \delta V_{i_m}}.$$

The terms with fourth derivatives of P_0 enter this expansion with coefficients equal to

$$b_{ijkl} - b_{ij} b_{kl} - b_{ik} b_{jl} - b_{il} b_{jk}; \tag{4}$$

if the hypothesis of a Gaussian relation between the fourth and second moments is adopted, they are equal to zero. (The sum of the squares of the coefficients of series (3) is equal to D, and if condition (2) is satisfied, the coefficients of the higher terms must decrease.) The terms with fifth derivatives enter series (3) with coefficients equal to the difference between the right- and left-hand sides of (1). For small amplitudes of the fluctuations the distribution P differs little from the Gaussian one, and one may expect that series (3) converges. In this case, by virtue of (3), one may expect that the range of applicability of the hypothesis that expression (4) is equal to zero is restricted to such small amplitudes of the fluctuations for which the third moments $b_{i_k i_l i_m}$ are small: for large values of $b_{i_k i_l i_m}$, approximating P by the segment consisting of the first- and second-order terms of series (3) leads to negative values of the probability density P. Approximation by the terms of the first three orders of series (3) is more accurate, and one may expect that hypothesis (1) is valid for a broader range of fluctuation-amplitude values and is suitable for analyzing the nonlinear mechanism of loss of stability of laminar flows and the onset of turbulence.

II. For a plane-parallel flow with mean-velocity gradient $\partial U_1/\partial x_2 = C(\mathbf{x}, t)$ ([1]) and with the scale of the initial turbulent fluctuations smaller than the scale of the averaged motion, the Navier–Stokes equations then yield the following system of equations, closed with account of (1):

$$\left(\frac{\partial}{\partial t}+U_k\frac{\partial}{\partial x_k}\right)\Phi_{ij}(\mathbf{k},\mathbf{x},t)-\frac{\partial U_k}{\partial x_l}k_k\frac{\partial}{\partial k_l}\Phi_{ij}+$$

$$+2\nu k^2\Phi_{ij}+\frac{\partial U_l}{\partial x_k}\left(\delta_{lj}-2\frac{k_lk_j}{k^2}\right)\Phi_{ik}+\frac{\partial U_l}{\partial x_k}\left(\delta_{li}-2\frac{k_lk_i}{k^2}\right)\Phi_{kj}-$$

$$-ik_k\Phi_{li,j}+ik_l\Phi_{i,lj}+i\frac{k_ik_lk_s}{k^2}\Phi_{ls,j}-i\frac{k_jk_lk_s}{k^2}\Phi_{i,ls}=0. \qquad (5)$$

Here it is denoted that

$$\Phi_{i,li}(\mathbf{k})=\int_{-\infty}^{\infty}\Phi_{ili}(\mathbf{k};\mathbf{k}')\,d\mathbf{k}' \quad \text{etc.}$$

$$\Phi_{ilj}(\mathbf{k}_{(1)};\mathbf{k}_{(2)})=\int_{-\infty}^{\infty}e^{i(\mathbf{k}_{(1)}\mathbf{r}_{(1)}+\mathbf{k}_{(2)}\mathbf{r}_{(2)})}\overline{\delta V_i(\mathbf{x}-\mathbf{r}_{(1)})\delta V_l(\mathbf{x})\delta V_j(\mathbf{x}+\mathbf{r}_{(2)})}\,d\mathbf{r}_{(1)}\,d\mathbf{r}_{(2)};$$

$$\frac{\partial}{\partial t}\Phi_{ijk}(\mathbf{k}_{(1)};\mathbf{k}_{(2)};\mathbf{x};t)+2\nu\left(k_{(1)}^2+k_{(2)}^2-\mathbf{k}_{(1)}\mathbf{k}_{(2)}\right)\Phi_{ijk}-$$

$$-\frac{\partial U_1}{\partial x_2}k_{1(1)}\frac{\partial}{\partial k_{2(1)}}\Phi_{ijk}-\frac{\partial U_1}{\partial x_2}k_{1(2)}\frac{\partial}{\partial k_{2(2)}}\Phi_{ijk}+\delta_{i1}\frac{\partial U_1}{\partial x_2}\Phi_{2jk}+$$

$$\begin{aligned}&+\delta_{j1}\frac{\partial U_1}{\partial x_2}\Phi_{i2k}+\delta_{k1}\frac{\partial U_1}{\partial x_2}\Phi_{ij2}-2\frac{k_{1(1)}k_{i(1)}}{k_{(1)}^2}\frac{\partial U_1}{\partial x_2}\Phi_{2jk}\\&\quad-\frac{2k_{k(2)}k_{1(2)}}{k_{(2)}^2}\frac{\partial U_1}{\partial x_2}\Phi_{ij2}-2\frac{(k_{j(1)}-k_{j(2)})(k_{1(1)}-k_{1(2)})}{(k_{(1)}-k_{(2)})^2}\frac{\partial U_1}{\partial x_2}\Phi_{i2k}\\&=i\{k_{l(1)}\Phi_{li,j,k}-(k_{l(1)}-k_{l(2)})\Phi_{i,lj,k}-k_{l(2)}\Phi_{i,j,lk}\\&\quad-(k_{l(1)}k_{s(1)}k_{i(1)}/k_{(1)}^2)\Phi_{sl,j,k}+(k_{l(2)}k_{s(2)}k_{k(2)}/k_{(2)}^2)\Phi_{i,j,ls}\\&\quad+((k_{l(1)}-k_{l(2)})(k_{s(1)}-k_{s(2)})(k_{j(1)}-k_{j(2)})/(k_{(1)}-k_{(2)})^2)\Phi_{i,ls,k}\};\end{aligned} \qquad (6)$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Phi_{iqkm}(k_{(1)};k_{(2)};k_{(3)};t)-\left(k_{1(1)}\frac{\partial}{\partial k_{2(1)}}+k_{1(2)}\frac{\partial}{\partial k_{2(2)}}+k_{1(3)}\frac{\partial}{\partial k_{2(3)}}\right)\Phi_{iqkm}\frac{\partial U_1}{\partial x_2}\\
&+\delta_{i1}(\partial U_1/\partial x_2)\Phi_{2qkm}+\delta_{j1}(\partial U_1/\partial x_2)\Phi_{i2km}+\delta_{k1}\frac{\partial U_1}{\partial x_2}\Phi_{iq2m}\\
&+\delta_{m1}(\partial U_1/\partial x_2)\Phi_{iqk2}+2\nu\,[k_{(1)}^2+k_{(2)}^2+k_{(3)}^2-k_{(1)}k_{(2)}-k_{(2)}k_{(3)}]\Phi_{iqkm}(\partial U_1/\partial x_2)\\
&-2(\partial U_1/\partial x_2)\,((k_{1(2)}-k_{1(3)})(k_{k(2)}-k_{k(3)})/(k_{(2)}-k_{(3)})^2)\Phi_{iq2m}\\
&-2(\partial U_1/\partial x_2)(k_{1(3)}k_{m(3)}/k_{(3)}^2)\Phi_{iqk2}-2(\partial U_1/\partial x_2)(k_{1(1)}k_{i(1)}/k_{(1)}^2)\Phi_{2qkm}\\
&-2(\partial U_1/\partial x_2)\,((k_{1(1)}-k_{1(2)})(k_{q(1)}-k_{q(2)})/(k_{(1)}-k_{(2)})^2)\Phi_{i2km}\\
&+i(k_{i(1)}k_{l(1)}k_{s(1)}/k_{(1)}^2)\Phi_{ls,q,k,m}\\
&-i((k_{q(1)}-k_{q(2)})(k_{s(1)}-k_{s(2)})(k_{i(1)}-k_{l(2)})/(k_{(1)}-k_{(2)})^2)\Phi_{i,ls,k,m}\\
&-i(k_{m(3)}k_{l(3)}k_{s(3)}/k_{(3)}^2)\Phi_{i,q,k,ls}\\
&-i((k_{k(2)}-k_{k(3)})(k_{l(2)}-k_{l(3)})(k_{s(2)}-k_{s(3)})/(k_{(2)}-k_{(3)})^2)\Phi_{i,q,ls,m}\\
&-ik_{j(1)}\Phi_{ij,q,k,m}-i(k_{j(2)}-k_{j(1)})\Phi_{i,jp,k,m}-i(k_{j(3)}-k_{j(2)})\Phi_{i,q,jk,m}\\
&+ik_{j(3)}\Phi_{i,q,k,jm}=0.
\end{aligned}
\tag{7}
$$

Let the initial disturbance be a wave with wave vector $\mathbf{k}_0$. Then the initial conditions for the system of equations (5)–(7), (1) will be

$$
\Phi_{ij}^0(\mathbf{k})=A(\delta_{ij}k^2-k_ik_j)[\delta(\mathbf{k}-\mathbf{k}_0)+\delta(\mathbf{k}+\mathbf{k}_0)];\qquad \Phi_{ijk}^0=0;\tag{8}
$$

Φ_{ijkl}^0 are expressed in terms of Φ_{ik}^0 in accordance with the Gaussian law.

The initial values differ from zero only at several points, and the solution of the system of first-order partial differential equations (5)–(7), (1) reduces to the solution of a system of first-order ordinary differential equations corresponding to the characteristics passing through these points. These equations are rapidly solved on an electronic computer. The solution shows the change in the wave amplitude with time.

The system of equations (5)–(7), (1) is essentially nonlinear. The solution depends substantially on the initial amplitude of the wave. Fig. 1 gives the results of a numerical solution on the M-20 computer of equations (5)–(7), (1), as well as of the analogous closed system of equations obtained by means of hypothesis (4), with initial conditions (8). For $A \ll 1$ the solution of both nonlinear systems passes into the solution of the linear system obtained from (5) by discarding the third moments. Curve 1 shows the ratio of the energy of the pulsations

Fig. 1

Figure 1: Fig. 1

$$B(t) = \int_{-\infty}^{\infty} (\Phi_{11}(\mathbf{k}, t) + \Phi_{22}(\mathbf{k}, t))\, d\mathbf{k}$$

at time t to the initial energy $B(0)$, for sufficiently small initial amplitudes, for which the solution does not leave the linear region ($B(0) = 10^{-5}$; $\mathrm{Re} = 10^4$; $k_1^0 = 0.1$; $k_2^0 = 2$). In this case the energy $B(t)$ (in accordance with (1)) decays as $t \to \infty$. Cri-

ve 2 presents the results of calculation under hypothesis (4), $A > A_{\mathrm{cr}}$ ($B(0) = 0.0074$; $\mathrm{Re} = 10^4$; $k_1^0 = 0.1$; $k_2^0 = 4$). As is seen from the figure, the quantity $B(t)$ takes negative values. This indicates the inadequacy of relation (4) for comparatively large amplitudes of the pulsations. Curve 3 presents the results of solving (5)–(7), (1) for $A > A_{\mathrm{cr}}$. For $A < A_{\mathrm{cr}}$, $B(t) \to 0$ as $t \to \infty$. For $A > A_{\mathrm{cr}}$, all waves that are multiples of k_0 are excited.

Fig. 1

The hypothesis of a Gaussian relation between the fourth and second moments, as is seen from Fig. 1, remains valid only over a short interval of time. Hypothesis (1) has a considerably longer, but likewise finite, interval of amplitudes over which it is applicable. The next (see series (3)) approximation–the hypothesis

$$b_{j_1 j_2 j_3 j_4 j_5 j_6} = \sum_{\substack{k<l;\ i=2,\dots,6;\\ p,m,q,s\neq k,l}} b_{j_k j_l}\, b_{j_p j_m j_q j_s}$$

has an even larger, but likewise finite, interval of values for which it is applicable.

In Fig. 1, curve 3 corresponds to $B(0) = 0.03$; $k_1^0 = 0.1$; $k_2^0 = 2$; $\mathrm{Re} = 10^4$. Curve 4 is the solution of (5)–(7), (1) for the same $k_1^0 = 0.1$, $k_2^0 = 2$, $B(0) = 0.73$, but $\mathrm{Re} = 10$. In this case $B(t)$ decreases as t increases.

The numerical solution makes it possible to determine the magnitude of the critical amplitude as a function of the wave number k_0 and the Reynolds number Re (here one may take $\mathrm{Re} = (\partial U_1/\partial x_2)(1/\nu k_0^2)$); $A_{\mathrm{cr}} = A_{\mathrm{cr}}(k_0; \mathrm{Re})$. As $\mathrm{Re} \to \infty$, the quantity $\min_{k_0} A_{\mathrm{cr}}(k_0; \mathrm{Re})$ tends to zero.

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Note: Figure translations are in progress. See original paper for figures.