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Abstract

Full Text

Mathematics

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Estimates in L_p of Solutions of Elliptic Equations with Discontinuous Coefficients Satisfying General Boundary Conditions and Conjugation Conditions

(Presented by Academician S. L. Sobolev on 26 IX 1962)

1°. In papers ^(1,2), devoted to the proof and use of energy inequalities for elliptic equations with discontinuous coefficients (a bibliography is also given there), inequalities in the norm L_2 were considered. In the present note analogous estimates are obtained in the norm L_p for arbitrary $p > 1$. The method of proof is close to that used in ⁽³⁾: first a formula is derived which gives an explicit representation of the solution of the "conjugation problem" for an elliptic equation with piecewise constant coefficients in the whole space, with a jump of the coefficients along the hyperplane $x_n = 0$, and then, with the aid of the Calderon-Zygmund theorem ⁽⁴⁾, the required inequalities are proved.

Let us note that for $p = 2$ the results given below do not completely cover the content of paper ⁽¹⁾, since in ⁽¹⁾ the systems of boundary conditions and conjugation conditions may be overdetermined; on the other hand, in the present paper boundary operators and conjugation operators of arbitrary order are considered, whereas in ⁽¹⁾ their orders are below the order $2m$ of the equation.

2°. Let G be a bounded domain of n -dimensional Euclidean space E_n with boundary Γ ; let G_1 be a subdomain of G with boundary γ , having no common points with Γ ; $G_2 = G \setminus \overline{G_1}$.* Introduce the direct sum of Sobolev spaces

$$W_p^l(G_1) + W_p^l(G_2) = W_p^l(G) = W_p^l \quad (l \geq 0 \text{ integer, } p > 1);$$

the norm in W_p^l will be denoted by

$$\|u\|_{W_p^l(G)} = \left(\sum_{|\mu| \leq l} \int_G |D^\mu u|^p dx \right)^{1/p}$$

or, more briefly, $\|u\|_l$.

Every function $u \in W_p^l$ can be represented in the form $u(x) = u_1(x) + u_2(x)$, where

$$u_r(x) = \begin{cases} u(x), & x \in G_r, \\ 0, & x \in G \setminus \overline{G_r} \end{cases} \quad (r = 1, 2).$$

If $x \in \gamma$, then $u_r(x)$ denotes the limiting value of $u(x)$ from the side of G_r .

Denote by $W_p^{l-1/p}(S)$ the space of functions φ defined on the boundary S of a certain domain Δ and which are the values on S of functions $v \in W_p^l(\Delta)$; the norm in this space is defined as follows:

$$\|\varphi\|_{l-1/p} \equiv \|\varphi\|_{W_p^{l-1/p}(S)} = \inf \|v\|_l,$$

where the infimum is taken over all $v \in W_p^l(\Delta)$ equal to φ on S (see (5,6,3)).

We prescribe an elliptic differential operator A with discontinuous complex coefficients

$$(Au)(x) = \begin{cases} (A^1u)(x), & x \in G_1, \\ (A^2u)(x), & x \in G_2, \end{cases} \quad (1)$$

where

$$A^r = \sum_{|\mu| \leq 2m} a_{\mu}^r(x) D^{\mu}, \quad x \in G_r \quad (r = 1, 2); \quad \mu = (\mu_1, \dots, \mu_n), \quad D^{\mu} = D_1^{\mu_1} \dots$$

* All that follows is also valid for a partition of G into a finite number of domains with surfaces having no corners.

... $D_n^{\mu_n}$, $D_k = \partial/\partial x_k$, as well as **boundary operators** (on Γ) and **conjugation operators** (on γ):

$$B_k^r = \sum_{|\mu| \leq m_k^r} b_{k\mu}^r(x) D^{\mu} \quad (2)$$

$$(r = 1, 2, 3; k = 1, \dots, i_r; i_1 = i_2 = 2m, i_3 = m; m_k^1 = m_k^2 = m_k),$$

where the complex functions $b_{k\mu}^1(x)$, $b_{k\mu}^2(x)$ are defined on γ , and $b_{k\mu}^3(x)$ on Γ . We denote the characteristic polynomials of the operators A^r , B_k^r , respectively, by

$$P^r(x, \xi) = \sum_{|\mu|=2m} a_{\mu}^r(x) \xi^{\mu}, \quad Q_k^r(x, \xi) = \sum_{|\mu|=m_k^r} b_{k\mu}^r(x) \xi^{\mu} \quad (\xi = (\xi_1, \dots, \xi_n))$$

a complex vector). The ellipticity condition means that $P^r(x, \xi) \neq 0$ in $\overline{G_r}$ ($r = 1, 2$) for every real vector $\xi \neq 0$. Let x be any fixed point on Γ (γ); consider the polynomial $P^r(\eta) = P^r(x, \tau + \eta\nu)$, where $\tau \neq 0$ is any real vector tangent to Γ (γ) at the point x , and ν is the normal direction to Γ (γ) at x . From ellipticity it follows that $P^r(\eta)$ has no real roots; we shall require that

the complex roots of these polynomials be distributed equally in the upper and lower half-planes (for $n \geq 3$ this always holds (7)).

Denote $l_1 = \max\{2m, m_k^r + 1\}$, and let $l \geq l_1$ be a fixed integer. We shall assume that Γ and γ are surfaces of class C^l , and that the coefficients $a_\mu^r(x) \in C^{l-2m}(\overline{G_r})$, $b_{k\mu}^r(x) \in C^{l-m_k^r}(\gamma)$ ($r = 1, 2$), $b_{k\mu}^3(x) \in C^{l-m_k^3}(\Gamma)$. Consider the following problem: to find in \overline{G} a function u for which

$$Au = f \quad \text{in } G \setminus \gamma;$$

$$[B_k u] = B_k^1 u_1 - B_k^2 u_2|_\gamma = \varphi_k, \quad k = 1, \dots, 2m; \quad (3)$$

$$B_j^3 u|_\Gamma = \psi_j, \quad j = 1, \dots, m,$$

where $f \in W_p^{l-2m}$, $\varphi_k \in W_p^{l-m_k-1/p}(\gamma)$, $\psi_j \in W_p^{l-m_j^3-1/p}(\Gamma)$.

Theorem 1. Let $u \in W_p^{l_1}$ be a solution of problem (3). A necessary and sufficient condition that there exist a constant $K > 0$, independent of f, φ_k, ψ_j , such that

$$\|u\|_l \leq K \left(\|f\|_{l-2m} + \sum_{k=1}^{2m} \|\varphi_k\|_{l-m_k-1/p} + \sum_{j=1}^m \|\psi_j\|_{l-m_j^3-1/p} + \|u\|_0 \right) \quad (4)$$

is that the operator A be elliptic and that the operators (2) cover it.*

Proof. The proof of sufficiency, by means of a partition of unity, reduces to the proof of local inequalities of type (4). Since for interior points and for points of the surface Γ such inequalities have been proved in (3), it is enough to restrict ourselves to points lying on the surface of discontinuity γ ; it is necessary to prove that for every point $x \in \gamma$ there exists a neighborhood U in E_n such that inequality (4) holds for all solutions $u \in W_p^{l_1}$ of problems (3) that vanish outside U . This is proved, as in (3), by constructing in the simplest case an explicit representation of the solution, which we shall consider in the following item.

3°. Let A^r, B_k^r ($r = 1, 2; k = 1, \dots, 2m$) be operators with constant coefficients, containing only terms of highest order; denote

* The definition of covering for the case of discontinuous coefficients is given in (1) (Definition 2).

for convenience $x = (x_1, \dots, x_{n-1})$, $y = x_n$; $\xi = (\xi_1, \dots, \xi_{n-1})$, $\eta = \xi_n$; $P^r(\xi, \eta)$, $Q_k^r(\xi, \eta)$ are the corresponding characteristic polynomials; let us consider the following problem:

Conjugation problem. Let $\varphi_k(x) \in C^\infty(E_{n-1})$ ($k = 1, \dots, 2m$) be finite functions; one seeks a function $u(x, y)$, infinitely differentiable in each of the half-planes $y > 0$, $y < 0$, and such that

$$Au = f,$$

$$[B_{ku}]_{y=0} = B_k^1 u(x, +0) - B_k^2 u(x, -0) = \varphi_k, \quad k = 1, \dots, 2m. \quad (5)$$

We first construct a solution of this problem for $f \equiv 0$. For real $\xi \neq 0$ put $P^r(\xi, \eta) = P^{r+}(\xi, \eta)P^{r-}(\xi, \eta)$, where all η -roots of $P^{r+}(P^{r-})$ lie in the upper (lower) half-plane. Let

$$P^{r\pm}(\xi, \eta) = \sum_{t=0}^m \alpha_t^{r\pm}(\xi) \eta^{m-t};$$

denote

$$P_s^{r\pm} = \sum_{t=0}^s \alpha_t^{r\pm}(\xi) \eta^{s-t} \quad (s = 0, \dots, m-1),$$

then (3)

$$\frac{1}{2\pi i} \int_{\Lambda^\pm} \frac{P_{m-p}^{r\pm}(\xi, \eta)}{P^{r\pm}(\xi, \eta)} \eta^{q-1} d\eta = \delta_{pq}, \quad p, q = 1, \dots, m; \quad r = 1, 2, \quad (6)$$

where $\Lambda^+(\Lambda^-)$ is a rectifiable contour enclosing all the η -roots of $P^{r+}(P^{r-})$.

For fixed $\xi \neq 0$, denote by

$$Q_k^r(\xi, \eta) = \sum_{q=1}^m \beta_{kq}^r(\xi) \eta^{q-1}$$

($r = 1, 2$) the remainder upon division of $Q_k^1(\xi, \eta)$ by $P^{1+}(\xi, \eta)$ (respectively Q_k^2 by P^{2-}). From the condition of joint covering (see (1)) it easily follows that

$$d(\xi) = \det \left\| (-1)^{r-1} \beta_{kq}^r(\xi) \right\|_{\substack{k=1, \dots, 2m, \\ q=1, \dots, m; \quad r=1, 2}} \neq 0;$$

if

$$\left\| \beta_r^{pj}(\xi) \right\|_{\substack{r=1, 2; \quad p=1, \dots, m, \\ j=1, \dots, 2m}}$$

is the matrix inverse to the determinant matrix $d(\xi)$, and

$$N_j^1(\xi, \eta) = \sum_{p=1}^m \beta_1^{pj} P_{m-p}^{1+}, \quad N_j^2(\xi, \eta) = \sum_{p=1}^m \beta_2^{pj} P_{m-p}^{2-} \quad (j = 1, \dots, 2m),$$

then from (6) it easily follows that

$$\frac{1}{2\pi i} \int_{\Lambda} \left(\frac{N_j^1 Q_k^1}{P^{1+}} - \frac{N_j^2 Q_k^2}{P^{2-}} \right) d\eta = \delta_{jk}, \quad j, k = 1, \dots, 2m, \quad (7)$$

where Λ is a contour in the complex plane enclosing all the roots of P^{1+} and P^{2-} .

We now introduce Poisson kernels, analogous to those constructed in (3), but defined in each of the half-spaces $y > 0$, $y < 0$:

$$K_k(x, y) = \begin{cases} K_k^1(x, y), & y > 0, \\ K_k^2(x, y), & y < 0, \end{cases}$$

where

$$K_k^1(x, y) = -\frac{1}{(2\pi i)^n (m_k - n + 1)!} \int_{|\xi|=1} d\omega_{\xi} \int_{\Lambda} \frac{N_k^1(\xi, \eta) (x\xi + y\eta)^{m_k - n + 1}}{P^{1+}(\xi, \eta)} \ln \frac{x\xi + y\eta}{i} d\eta, \quad m_k \geq n, \quad (8)$$

$$K_k^1(x, y) = (-1)^{n-1-m_k} \frac{(n - m_k - 2)!}{(2\pi i)^n} \times \int_{|\xi|=1} d\omega_{\xi} \int_{\Lambda} \frac{N_k^1(\xi, \eta) d\eta}{P^{1+}(\xi, \eta) (x\xi + y\eta)^{n-1-m_k}}, \quad 0 \leq m_k < n - 1; \quad (9)$$

K_k^2 are defined analogously, but with N_k^1 replaced by N_k^2 and P^{1+} by P^{2-} . Here $d\omega_{\xi}$ is the surface element of the sphere $|\xi| = 1$; Λ is a contour in the complex plane containing inside it all η -roots of P^{1+} and P^{2-} for all $|\xi| = 1$ and not enclosing the origin (such a contour exists; see (3)).

Theorem 2. The function

$$u(x, y) = \sum_{k=1}^{2m} \int K_k(x - x', y) \varphi_k(x') dx' = \sum_{k=1}^{2m} K_k * \varphi_k \quad (10)$$

is a solution of the conjugation problem (5) with $f \equiv 0$.

Proof is carried out by the same method as in (3), and is based on relation (7).

Using the solution formula (10) thus obtained, one can, with the aid of the fundamental solution for the operator A , construct a representation formula for the solution of the nonhomogeneous problem (5). Since the formula obtained has the same structure as the corresponding formula in (3), one may apply to it the Calderón-Zygmund theorem (Theorem 2 of (4)) and Theorem 3.4 (c) of (3), and obtain the required estimates.

In exactly the same way, from the representation formula found, estimates of Schauder type are derived with the aid of Theorem 3.4 (a) of (3).

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Note: Figure translations are in progress. See original paper for figures.

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