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Abstract

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MATHEMATICS

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ON THE SUM OF DIFFERENCES OF EIGENVALUES OF TWO SINGULAR STURM-LIOUVILLE OPERATORS

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1. A formula for the sum of differences of the eigenvalues of two regular Sturm-Liouville operators was first obtained by I. M. Gelfand and B. M. Levitan (^{1}). Subsequently, an analogous formula was proved in (^{2}) for the case of two singular Sturm-Liouville operators with discrete spectra, differing from one another only by a finite perturbation.

In the present paper we study the sum of differences of the eigenvalues of two singular Sturm-Liouville operators differing from one another by boundary conditions and a finite perturbation. An analogue of the formula mentioned above is obtained, and several necessary conditions are proved in order that two sequences of numbers $\{\lambda_n\}$ and $\{\mu_n\}$ be the eigenvalues of one singular Sturm-Liouville equation, but with different boundary conditions.

2. Consider the differential equation

$$l[y] \equiv -y'' + q(x)y = \lambda y \tag{1}$$

and the boundary conditions

$$y'(0) - h_1 y(0) = 0; \tag{2}$$

$$y'(0) - h_2 y(0) = 0, \tag{3}$$

where $0 \leq x < \infty$; $q(x)$ is a real function summable on every interval $(0, b)$ $(b < \infty)$; h_1 and h_2 are real numbers, with $h_1 \neq h_2$. Next define the boundary condition at the point b:

$$y'(b)\sin\beta + y(b)\cos\beta = 0. \tag{4}$$

It is obvious that for any finite b the spectra of the boundary-value problems (1), (2), (4) $[L_{h_1}(b)]$ and (1), (3), (4) $[L_{h_2}(b)]$ are bounded below and discrete. Therefore they can be numbered in increasing order. Let $\lambda_{-p}(b) < \lambda_{-p+1}(b) < \cdots < \lambda_0^{(b)} < \lambda_1(b) \cdots$ and $\mu_{-p}(b) < \mu_{-p+1}(b) < \cdots$ be the eigenvalues (here $\lambda_0(b) \leq 0$, $\lambda_1(b) > 0$), and let $\{\varphi_{n,b}(x)\}$ and $\{\psi_{n,b}(x)\}$ be the corresponding orthonormal eigenfunctions.

Lemma 1. *For every integer $N \geq 0$ the inequalities*

$$(h_2 - h_1)\sum_{n=-p}^{N} \psi_{n,b}^2(0) \leq \sum_{n=-p}^{N} \{\mu_n(b) - \lambda_n(b)\} \leq (h_2 - h_1)\sum_{n=-p}^{N} \varphi_{n,b}^2(0). \tag{5}$$

Proof. By Parseval' s equality,

$$\mu_n(b) = (l[\psi_{n,b}], \psi_{n,b}) = \sum_{k=-p}^{\infty} (l[\psi_{n,b}], \varphi_k) u_{nk},$$

where

$$u_{nk} = (\varphi_{k,b}, \psi_{n,b}) = \int_0^b \varphi_{k,b}(x)\psi_{n,b}(x)\,dx.$$

Integrating by parts, we obtain

$$(l[\psi_{n,b}], \varphi_{k,b}) = (h_2 - h_1)\psi_{n,b}(0)\varphi_{k,b}(0) + \lambda_k u_{nk}.$$

Therefore

$$\mu_n(b) = \sum_{k=-p}^{\infty} \lambda_k(b) u_{nk}^2 + (h_2 - h_1)\psi_{n,b}^2(0).$$

Hence, just as in (^{2}), it is not difficult to obtain that

$$\sum_{n=-p}^{N} \mu_n(b) \geq \sum_{n=-p}^{N} \lambda_n(b) + (h_2 - h_1)\sum_{n=-p}^{N} \psi_{n,b}^2(0).$$

This proves the left-hand side of inequality (5). The right-hand side of inequality (5) is proved analogously. The lemma is proved.

Let us now find equations for determining the numbers $\lambda_n(b)$ and $\mu_n(b)$, $n = -p, -p+1, \ldots$. To this end, consider the functions $\xi(x, \lambda)$ and $\theta(x, \lambda)$, which are solutions of equation (1) and satisfy the initial conditions:

$$\xi(0, \lambda) = 1, \qquad \xi'(0, \lambda) = h_1; \tag{6}$$

$$\theta(0, \lambda) = 1, \qquad \theta'(0, \lambda) = h_2. \tag{7}$$

Let $f(x, \lambda) = \theta(x, \lambda) + l^b_{h_1}(\lambda)\xi(x, \lambda)$ satisfy the boundary condition (4). Then

$$l^b_{h_1}(\lambda) = -\frac{\theta'(b, \lambda)\sin\beta + \theta(b, \lambda)\cos\beta}{\xi'(b, \lambda)\sin\beta + \xi(b, \lambda)\cos\beta}. \tag{8}$$

It is obvious that the eigenvalues of the operators $L_{h_1}(b)$ and $L_{h_2}(b)$ coincide respectively with the poles and zeros of the function $l^b_{h_1}(\lambda)$.

As is known (see (^{3}), p. 35), one can choose such sequences of numbers $b_k \to \infty$ and $\beta_k \to \beta_0$ as $k \to \infty$ that the sequence of functions $l^{b_k}_{h_1}(\lambda)$ has a limit $l_{h_1}(\lambda)$ as $k \to \infty$. In this case it is clear that we obtain two self-adjoint extensions of the differential equation (1). Denote the resulting operators by L_{h_1} and L_{h_2}. Suppose that the function $q(x)$ is such that any self-adjoint extension of equation (1) has a discrete spectrum with limit point at infinity. Then it is obvious that, for any fixed $n \geq 0$ and as $k \to \infty$, the sequences of numbers $\{\lambda_n(b_k)\}$ and $\{\mu_n(b_k)\}$ tend to the eigenvalues of the operators L_{h_1} and L_{h_2}, respectively. Denote $\lambda_n = \lim_{k\to\infty} \lambda_n(b_k)$, $\mu_n = \lim_{k\to\infty} \mu_n(b_k)$, and the corresponding normalized eigenfunctions of the operators L_{h_1} and L_{h_2} by $\varphi_n(x)$ and $\psi_n(x)$. In what follows, everywhere, without loss of generality, assume that $h_2 > h_1$. Then it is known that $\lambda_n(b) < \mu_n(b) < \lambda_{n+1}(b)$ and $\lambda_n < \mu_n < \lambda_{n+1}$. Therefore, from inequalities (5) it follows that, for any finite λ,

$$(h_2 - h_1) \sum_{\mu_n(b)<\lambda} \psi^2_{n,b}(0) \leq \sum_{\mu_n(b)<\lambda} \{\mu_n(b) - \lambda_n(b)\} \leq$$

$$\leq \sum_{\lambda_n(b)<\lambda} \{\mu_n(b) - \lambda_n(b)\} \leq (h_2 - h_1) \sum_{\lambda_n(b)<\lambda} \varphi^2_{n,b}(0). \tag{9}$$

Replacing now in these inequalities b by b_k and passing to the limit as $k \to \infty$, we obtain the following lemma:

Lemma 2. *For any finite λ the inequalities*

$$(h_2-h_1) \sum_{\mu_n<\lambda} \psi^2_n(0) \leq \sum_{\mu_n<\lambda} (\mu_n-\lambda_n) \leq\leq \sum_{\lambda_n<\lambda} (\mu_n-\lambda_n) \leq (h_2-h_1) \sum_{\lambda_n<\lambda} \varphi^2_n(0). \tag{10}$$

From these inequalities and from the estimate

$$\sum_{\lambda_n<\lambda} \varphi_n^2(0) = \rho_{h_1}(\lambda) - \rho_{h_1}(-\infty) \quad (\text{see } (^4,^5\,))$$

it obviously follows

Corollary. The series of the sum of the differences of the negative eigenvalues of the operators L_{h_1} and L_{h_2} converges. (In the case where the operators L_{h_1} and L_{h_2} are bounded below, the series becomes a finite sum.)

Remark 1. If no boundary condition need be prescribed at infinity, then $l_{h_1}^b(\lambda)$, $\mu_n(b)$, and $\lambda_n(b)$ converge to limits as $b \to \infty$.

Theorem 1. *The following formulas hold:*

$$\lim_{\lambda\to\infty} \left\{ \sum_{\mu_n<\lambda} (\mu_n - \lambda_n) - \frac{2}{\pi}(h_2 - h_1)\sqrt{\lambda} \right\} = -\frac{1}{2}(h_2^2 - h_1^2); \tag{11}$$

$$\lim_{\lambda\to\infty} \left\{ \sum_{\lambda_n<\lambda} (\mu_n - \lambda_n) - \frac{2}{\pi}(h_2 - h_1)\sqrt{\lambda} \right\} = -\frac{1}{2}(h_2^2 - h_1^2). \tag{12}$$

Proof. Denote by $\rho_{h_1}(\lambda)$ the spectral function of the operator L_{h_1} (for the definition of the function $\rho_{h_1}(\lambda)$, see paper $(^4)$). Then for $\lambda > 0$

$$\sum_{\lambda_n<\lambda} \varphi_n^2(0) = \rho_{h_1}(\lambda) - \rho_{h_1}(-\infty). \tag{13}$$

From the asymptotic formula for the spectral function and formula (13) it follows $(^4,^5\,)$ that, for large $\lambda > 0$,

$$\sum_{\lambda_n<\lambda} \varphi_n^2(0) = \frac{2}{\pi}\sqrt{\lambda} - h_1 + o(1). \tag{14}$$

Similarly,

$$\sum_{\mu_n<\lambda} \psi_n^2(0) = \frac{2}{\pi}\sqrt{\lambda} - h_2 + o(1). \tag{15}$$

Therefore from inequalities (10) it follows that

$$\frac{2}{\pi}(h_2 - h_1)\sqrt{\lambda} - h_2(h_2 - h_1) + o(h_2 - h_1) \le \sum_{\mu_n<\lambda} (\mu_n - \lambda_n) \le$$

$$\le \frac{2}{\pi}(h_2 - h_1)\sqrt{\lambda} - h_1(h_2 - h_1) + o(h_2 - h_1). \tag{16}$$

Let us now divide the interval $[h_1, h_2]$ into p equal parts and consider the operators $\widetilde{L}_{H_q}$ $(q = 0, 1, \dots, p)$ $(\widetilde{L}_{H_0} = L_{h_1}, \widetilde{L}_{H_p} = L_{h_2})$, generated by the differential equation (1) and by the boundary conditions $y_i'(0) - H_{qy}i(0) = 0$, respectively. (If a boundary condition at infinity is needed, then for all the operators it is one and the same.) Here we choose $H_q = h_1 + q\Delta h$, $\Delta h = \dfrac{h_2 - h_1}{p}$. Let the operator $\widetilde{L}_q$ have discrete eigenvalues $\{\lambda_{n,q}\}$, numbered in increasing order. Then, by inequalities (16)

$$-H_q \cdot \Delta h + o(\Delta h) \leqslant \sum_{\lambda_{n,q}<\lambda} (\lambda_{n,q} - \lambda_{n,q-1}) - \frac{2\Delta h}{\pi}\sqrt{\lambda} \leqslant -H_{q-1} \cdot \Delta h + o(\Delta h)$$

for $q = (p, p-1, \dots, 1)$. Adding these inequalities, we obtain

$$-\sum_{q=0}^{p} H_q \cdot \Delta h + o(1) \leqslant \sum_{\mu_n<\lambda} (\mu_n - \lambda_n) - \frac{2}{\pi}(h_2 - h_1)\sqrt{\lambda} \leqslant -\sum_{q=0}^{p-1} H_q \cdot \Delta h + o(1).$$

Let now $p \to \infty$ and, consequently, $\Delta h \to 0$. Then in the limit we obtain

$$\sum_{\mu_n<\lambda} (\mu_n - \lambda_n) - \frac{2}{\pi}(h_2 - h_1)\sqrt{\lambda} = -\frac{1}{2}(h_2^2 - h_1^2) + o(1), \tag{17}$$

which proves formula (11). Formula (12) is proved in an analogous way. The theorem is proved.

Using now formulas (11) and (12), it is not difficult to prove the following theorem:

Theorem 2. Let $\lambda_n \neq 0$, $\mu_n \neq 0$, $n = 0, \pm 1, \pm 2, \dots$. Then the series* converge absolutely

$$S_1 = \sum_{n=-\infty}^{\infty} \frac{\mu_n - \lambda_n}{\mu_n}; \tag{18}$$

$$S_2 = \sum_{n=-\infty}^{\infty} \frac{\mu_n - \lambda_n}{\lambda_n}. \tag{19}$$

3. Along with equation (1), consider the equation

$$-y'' + \{q(x) + p(x)\}y = \lambda y, \quad 0 \leqslant x < \infty, \tag{20}$$

where $p(x)$ is a finite function. Suppose the function $q(x)$ is such that the boundary-value problem (1)–(2) has a discrete spectrum bounded below, $\lambda_1 < \lambda_2 < \cdots$.** Then the boundary-value problem (20)–(2) also has a discrete spectrum bounded below, $\mu_1 < \mu_2 < \cdots$.

Theorem 3. If $p(x)$ is differentiable in a neighborhood of zero and

$$\int P(x)\,dx = 0,$$

then

$$\lim_{\lambda\to\infty}\left\{\sum_{\mu_n<\lambda}(\mu_n-\lambda_n)-\frac{2}{\pi}(h_2-h_1)\sqrt{\lambda}\right\}=-\frac{1}{2}(h_2^2-h_1^2)+\frac{P(0)}{4}. \tag{21}$$

Proof. Denote by $\gamma_1 < \gamma_2 < \cdots$ the eigenvalues of the boundary-value problem (20)–(1). Then, by (2), for large λ

$$\sum_{\mu_n<\lambda}(\gamma_n-\lambda_n)=-\frac{P(0)}{4}+o(1). \tag{22}$$

On the other hand, by Theorem 1, for large λ,

$$\sum_{\mu_n<\lambda}(\mu_n-\gamma_n)-\frac{2}{\pi}(h_2-h_1)\sqrt{\lambda}=-\frac{1}{2}(h_2^2-h_1^2)+o(1). \tag{23}$$

Summing equalities (22) and (23) as $\lambda \to \infty$, we obtain formula (21), which proves the theorem.

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* If the spectra are bounded below, then $\lambda_n = 0$, $\mu_n = 0$ for sufficiently large negative n.

** One may dispense with the conditions that the spectrum be bounded below.