ON THE LINES OF FORCE OF THE MAGNETIC FIELD OF HELICAL CURRENTS FLOWING ON THE SURFACE OF A TORUS

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Abstract

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ON THE LINES OF FORCE OF THE MAGNETIC FIELD OF HELICAL CURRENTS FLOWING ON THE SURFACE OF A TORUS

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The present note is devoted to an investigation of the behavior of the lines of force of the magnetic field of helical currents flowing on the surface of a torus. This problem is of great importance in the theory of the stellarator. The results of the present investigation make it possible to assert that the asymmetry of the magnetic field under consideration is one of the causes of the appearance of protuberances on the surface of the plasma. The note also shows that, by means of an additional vertically directed magnetic field of a certain magnitude, the indicated effect of asymmetry of the helical magnetic field can be weakened to a considerable extent.

Thus, let the surface of the torus be obtained by rotating the circle $(x-R_0)^2+z^2=a^2$ about the z-axis ($R_0>a$). Introduce in the region bounded by the surface of the torus a toroidal system of coordinates (r,φ,ψ) by means of the relations: $x=(R_0+r\cos\varphi)\cos\psi$, $y=-(R_0+r\cos\varphi)\sin\psi$, $z=r\sin\varphi$. Suppose further that along the helical line $r=a$, $\varphi=l\psi+\varphi_0$, in the direction of increasing ψ, there flows a constant current I (l—an integer). To find the magnetic field created by this current we shall use the Biot-Savart law. By elementary calculations we find that the expressions for the components of the vector $\mathbf{H}$ can be written in the form

$$H_r=-\frac{I}{c}\int_0^{2\pi}[-alR_0\cos(\theta-\varphi_0+l\sigma)\sin\sigma+aR_a\sin(\theta-\varphi_0+l\sigma)\cos\sigma-$$

$$-a^2l\cos\varphi\sin\sigma+2R_a^2\sin\varphi\sin^2\sigma/2\Big]\,R^{-3}d\sigma,$$

$$H_\varphi=-\frac{I}{c}\int_0^{2\pi}[alR_0\sin(\theta-\varphi_0+l\sigma)\sin\sigma+aR_a\cos(\theta-\varphi_0+l\sigma)\cos\sigma-$$

$$-rR_a\cos\sigma+a^2l\sin\varphi\sin\sigma+2R_a^2\cos\varphi\sin^2\sigma/2-$$

$$-alr\sin(\varphi-\theta+\varphi_0-l\sigma)\sin\sigma]\,R^{-3}d\sigma, \tag{1}$$

$$H_\psi=-\frac{I}{c}\int_0^{2\pi}[-a^2l+alr\cos(\theta-\varphi_0+l\sigma)-rR_a\sin\varphi\sin\sigma+$$

$$+aR_a\sin(\varphi-\theta+\varphi_0-l\sigma)\sin\sigma+2alR_0\cos(\varphi-\theta+\varphi_0-l\sigma)\sin^2\sigma/2+$$

$$+2a^2l\sin^2\sigma/2-2alr\sin\varphi\sin(\varphi-\theta+\varphi_0-l\sigma)\sin^2\sigma/2\Big]\,R^{-3}d\sigma,$$

where $\theta=\varphi-l\psi$, $R_a=R_0+a\cos(\varphi-\theta+\varphi_0-l\sigma)$, and $R^2=4R_a(R_0+r\cos\varphi)\sin^2\sigma/2+a^2+r^2-2ar\cos(\theta-\varphi_0+l\sigma)$.

Let us examine the asymptotics of expressions (1) as $\frac{a}{R_0}\to 0$. Setting

$$r=a\rho,\qquad \psi=\frac{a}{R_0}z,\qquad \sigma=\frac{a}{R_0}\zeta,\qquad \alpha=\frac{al}{R_0}\quad\text{and}\quad \varepsilon=\frac{a}{R_0},$$

we readily obtain that, as $\varepsilon\to 0$, expressions (1) tend respectively to the following:

$$H_\rho=-\frac{I}{ac}\int_{-\infty}^{\infty}[-\alpha\zeta\cos(\theta-\varphi_0+\alpha\zeta)+\sin(\theta-\varphi_0+\alpha\zeta)]\,R^{-3}d\zeta,$$

$$H_\varphi=-\frac{I}{ac}\int_{-\infty}^{\infty}[\alpha\zeta\sin(\theta-\varphi_0+\alpha\zeta)+\cos(\theta-\varphi_0+\alpha\zeta)-\rho]\,R^{-3}d\zeta, \tag{2}$$

$$H_z=-\frac{I}{ac}\int_{-\infty}^{\infty}[-\alpha+\alpha\rho\cos(\theta-\varphi_0+\alpha\xi)]\,R^{-3}\,d\xi, \tag{2}$$

where $\theta=\varphi-\alpha z$, $\quad R^2=\xi^2+1+\rho^2-2\rho\cos(\theta-\varphi_0+\alpha\xi)$.

As was to be expected, the expressions written down coincide with the expressions for the components of the magnetic field obtained by means of a helical current flowing over the surface of a straight circular cylinder. This fact suggests that the answer to the question of the arrangement of the lines of force of the magnetic field of helical currents flowing over the surface of a torus, for small values of the ratio a/R_0, can be obtained by studying the arrangement of the lines of force of the magnetic field of helical currents flowing over the surface of a cylinder, upon which a specially chosen perturbation has been superposed.

In solving this problem we shall consider a system of $2n$ currents, of real interest, forming two families: the currents of the first family flow in the direction of increasing ψ along the helices $r = a,\ \varphi = l\psi + 2k\pi/n$ $(k = 1, 2, \dots, n)$, and the currents of the second family flow in the opposite direction along the helices $r = a,\ \varphi = l\psi + (2k-1)\pi/n$ $(k = 1, 2, \dots, n)$. The expressions for the components of the magnetic field obtained in this way are obtained by means of an obvious combination of expressions of the form (1). It follows directly from this that the magnetic field obtained will be periodic in θ, with period $2\pi/n$. Further, from what was said above it follows that the expressions for the components of this magnetic field can be represented in the form:

$$\begin{aligned} H_r(r, n\theta, \varphi, \varepsilon) &= H_r^0(r, n\theta) + \varepsilon H_r^1(r, n\theta, \varphi, \varepsilon), \\ H_\varphi(r, n\theta, \varphi, \varepsilon) &= H_\varphi^0(r, n\theta) + \varepsilon H_\varphi^1(r, n\theta, \varphi, \varepsilon), \\ H_\psi(r, n\theta, \varphi, \varepsilon) &= H_\psi^0(r, n\theta) + \varepsilon H_\psi^1(r, n\theta, \varphi, \varepsilon), \end{aligned} \tag{3}$$

where $H_r^0(r, n\theta)$, $H_\varphi^0(r, n\theta)$, and $H_\psi^0(r, n\theta)$ denote the corresponding combinations of expressions of the form (2). From these expressions it follows easily that $H_r^0(r, n\theta)$ is an odd function of θ, while $H_\varphi^0(r, n\theta)$ and $H_\psi^0(r, n\theta)$ are even functions of θ. Using further expressions (1), it is not difficult to verify that, as $\varepsilon \to 0$, the expressions for $H_r^1(r, n\theta, \varphi, \varepsilon)$, $H_\varphi^1(r, n\theta, \varphi, \varepsilon)$, and $H_\psi^1(r, n\theta, \varphi, \varepsilon)$ tend respectively to the following:

$$\begin{aligned} H_r^1 = &-\frac{I}{ac}\int_{-\infty}^{\infty}\sum_{m=1}^{2n}(-1)^m\big[\cos(\varphi-\theta+m\pi/n-\alpha\xi)\sin(\theta-m\pi/n+\alpha\xi) \\ &\qquad -\alpha\xi\cos\varphi+\tfrac{1}{2}\xi^2\sin\varphi\big]R_m^{-3}\,d\xi \\ &+\frac{3I}{2ac}\int_{-\infty}^{\infty}\sum_{m=1}^{2n}(-1)^m\big[\rho\cos\varphi+\cos(\varphi-\theta+m\pi/n-\alpha\xi)\big] \\ &\qquad \times\big[-\alpha\xi\cos(\theta-m\pi/n+\alpha\xi)+\sin(\theta-m\pi/n+\alpha\xi)\big]\xi^2R_m^{-5}\,d\xi, \\ H_\varphi^1 = &-\frac{I}{ac}\int_{-\infty}^{\infty}\sum_{m=1}^{2n}(-1)^m\big[\cos(\varphi-\theta+m\pi/n-\alpha\xi)\cos(\theta-m\pi/n+\alpha\xi) \\ &\qquad +\alpha\xi\sin\varphi+\tfrac{1}{2}\xi^2\cos\varphi-\rho\cos(\varphi-\theta+m\pi/n-\alpha\xi) \\ &\qquad -\alpha\rho\xi\sin(\varphi-\theta+m\pi/n-\alpha\xi)\big]R_m^{-3}\,d\xi \\ &+\frac{3I}{2ac}\int_{-\infty}^{\infty}\sum_{m=1}^{2n}(-1)^m\big[\rho\cos\varphi+\cos(\varphi-\theta+m\pi/n-\alpha\xi)\big] \\ &\qquad \times\big[\alpha\xi\sin(\theta-m\pi/n+\alpha\xi)+\cos(\theta-m\pi/n+\alpha\xi)-\rho\big]\xi^2R_m^{-5}\,d\xi, \end{aligned} \tag{4}$$

$$H_\psi^1 = -\frac{I}{ac}\int_{-\infty}^{\infty}\sum_{m=1}^{2n}(-1)^m[-\rho\xi\sin\varphi + \xi\sin(\varphi-\theta+m\pi/n-\alpha\xi)+$$

$$+\frac{1}{2}\alpha\xi^2\cos(\varphi-\theta+m\pi/n-\alpha\xi)]R_m^{-3}\,d\xi + \frac{3I}{2ac}\int_{-\infty}^{\infty}\sum_{m=1}^{2n}(-1)^m[\rho\cos\varphi+$$

$$+\cos(\varphi-\theta+m\pi/n-\alpha\xi)][-\alpha+\alpha\rho\cos(\theta-m\pi/n+\alpha\xi)]\xi^2R_m^{-5}\,d\xi,$$

where $R_m^2 = \xi^2+1+\rho^2-2\rho\cos(\theta-m\pi/n+\alpha\xi)$. From expressions (4) it follows that the equalities hold:

$$\begin{aligned} H_r^1(r,n\theta,\varphi,0) &= H_{r,1}^1(r,n\theta)\sin\varphi + H_{r,2}^1(r,n\theta)\cos\varphi,\\ H_\varphi^1(r,n\theta,\varphi,0) &= H_{\varphi,1}^1(r,n\theta)\sin\varphi + H_{\varphi,2}^1(r,n\theta)\cos\varphi,\\ H_\psi^1(r,n\theta,\varphi,0) &= H_{\psi,1}^1(r,n\theta)\sin\varphi + H_{\psi,2}^1(r,n\theta)\cos\varphi, \end{aligned} \tag{5}$$

where $H_{r,1}^1(r,n\theta)$, $H_{\varphi,2}^1(r,n\theta)$, and $H_{\psi,2}^1(r,n\theta)$ are even functions of θ, while $H_{r,2}^1(r,n\theta)$, $H_{\varphi,1}^1(r,n\theta)$, and $H_{\psi,1}^1(r,n\theta)$ are odd functions of θ.

Let now, in the region bounded by the surface of the torus, in addition to the helical magnetic field under consideration, there be a longitudinal magnetic field with components $H_r = H_\varphi = 0$, $H_\psi = H_0R_0/(R_0+r\cos\varphi)$. Then the equations of the lines of force of the total magnetic field can be written in the form:

$$\frac{dr}{d\psi} = \frac{(R_0+r\cos\varphi)^2H_r}{H_0R_0+(R_0+r\cos\varphi)H_\psi},$$

$$\frac{d\theta}{d\psi} = \frac{(R_0+r\cos\varphi)^2H_\varphi - rl[H_0R_0+(R_0+r\cos\varphi)H_\psi]}{rH_0R_0+r(R_0+r\cos\varphi)H_\psi}, \tag{6}$$

where $\theta = \varphi - l\psi$. Following the idea stated above, we make in system (6) the substitution: $r = a\rho$, $\psi = \frac{a}{R_0}z$, $\alpha = \frac{al}{R_0}$, and $\varepsilon = \frac{a}{R_0}$. As a result of the substitution, system (6) takes the form

$$\frac{d\rho}{dz} = \frac{[1+\varepsilon\rho\cos(\theta+\alpha z)]^2H_r}{H_0+[1+\varepsilon\rho\cos(\theta+\alpha z)]H_\psi},$$

$$\frac{d\theta}{dz} = \frac{[1+\varepsilon\rho\cos(\theta+\alpha z)]^2H_\varphi - \alpha\rho H_0 - \alpha\rho[1+\varepsilon\rho\cos(\theta+\alpha z)]H_\psi}{\rho H_0+\rho[1+\varepsilon\rho\cos(\theta+\alpha z)]H_\psi}, \tag{7}$$

Fig. 1

Figure 1: Fig. 1

where $\theta = \varphi - \alpha z$. The behavior of the lines of force of system (7) for $\varepsilon = 0$ is well known (see, for example, (1), Ch. VIII). It is necessary to note only the following detail. Depending on the signs of α and H_0, the vertices of the cylindrical n-gonal surface (the separatrix), separating regions with identical lines of force, in the space (r, θ, z) lie either on the half-planes $\theta = (2k-1)\pi/n$, or on the half-planes $\theta = 2k\pi/n$ $(k = 1, 2, \dots, n)$. The first case will occur when α and H_0 have the same signs, the second when α and H_0 have different signs. Since the second case is in principle no different from the first, in all subsequent reasoning we shall assume that α and H_0 have the same signs.

Fig. 1

To each lateral face of the cylindrical surface under consideration (we shall call them branches of the separatrix) we assign a number coinciding with the number of the current flowing opposite it, and we shall see what happens to each branch of the separatrix for small $\varepsilon \neq 0$. It follows from the general theory (see (2), § 5) that for small $\varepsilon \neq 0$ each of the branches of the separatrix splits into two related branches; the qualitative disposition of the intersections of these branches by the plane

$z = z_0$ coincides with that shown in Fig. 4 of my note[3]. To characterize the size and shape of the gap between related branches of the separatrix, let us consider their sections by the half-planes $\theta = 2k\pi/n$ $(k = 1, 2, \dots, n)$. The picture obtained in one of the sections is shown in Fig. 1. It is obvious that it will be periodic in z with period $2\pi/\alpha$. On those portions where the solid curve lies above the dashed one, the lines of force enter the region bounded by the branches of the separatrix; conversely, on those portions where the solid curve lies below the dashed one, the lines of force go out.

Let $\Delta_\varepsilon^k(z_0)$ be the distance between the points of the solid and dashed curves lying on the straight line $z = z_0$ (the index k refers to the related branches of the separatrix under consideration). Direct calculation shows that $\Delta_\varepsilon^k(z_0)$ has the form

$$\Delta_\varepsilon^k(z_0) = \varepsilon \Delta \sin(\alpha z_0 + 2k\pi/n) + \dots, \tag{8}$$

where the dots denote terms of higher order of smallness, and Δ does not depend on z_0. The following equality holds:

$$\Delta = K \int_{-\infty}^{\infty} \{[H_\varphi^0 - \alpha\rho(H_0 + H_\psi^0)][H_{r,1}^1 \cos(\theta + \alpha z) -$$

$$-H^1_{r,2}\sin(\theta+\alpha z)]+H^0_r[-(H^1_{\varphi,1}-\alpha\rho H^1_{\psi,1})\cos(\theta+\alpha z)+$$

$$+(H^1_{\varphi,2}-\alpha\rho H^1_{\psi,2})\sin(\theta+\alpha z)]\}(H_0+H^0_\psi)^{-1}\,dz+$$

$$+K\int_{-\infty}^{\infty}\alpha\rho^2 H^0_r(2H_0+H^0_\psi)(H_0+H^0_\psi)^{-1}\sin(\theta+\alpha z)\,dz, \qquad (9)$$

where the integration is carried out along the line of force $\rho=\rho(z)$, $\theta=\theta(z)$ of system (7) for $\varepsilon=0$, lying on the n-th branch of the separatrix and satisfying the condition $\theta(0)=0$; the constant K is equal to the value of the expression $-[H^0_\varphi-\alpha\rho(H_0+H^0_\psi)]^{-1}$ at the point $(\rho=\rho(0),\theta=0)$. The first of the integrals written above owes its origin to the helical field, the second to the longitudinal field.

Suppose now that, in the region bounded by the surface of the torus, a weak vertically directed magnetic field is added to the magnetic fields under consideration, with components $H_r=\dfrac{aH_1}{R_0}\sin\varphi$, $H_\varphi=\dfrac{aH_1}{R_0}\cos\varphi$, $H_\psi=0$. Under these conditions an additional term of the form $\varepsilon\delta H_1\sin(\alpha z_0+2k\pi/n)$ will appear in the expression for the function $\Delta^k_\varepsilon(z_0)$, where δ does not depend on z_0 and can be determined from the equality

$$\delta=K\int_{-\infty}^{\infty}\{[H^0_\varphi-\alpha\rho(H_0+H^0_\psi)]\cos(\theta+\alpha z)+$$

$$+H^0_r\sin(\theta+\alpha z)\}(H_0+H^0_\psi)^{-1}\,dz. \qquad (10)$$

The integration in equality (10) is carried out along the same line of force as in equality (9). Thus, by choosing H_1 so that the equality $\Delta+\delta H_1=0$ is satisfied, we thereby achieve a substantial reduction of the gap between related branches of the separatrix. This fact may have practical significance.

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Note: Figure translations are in progress. See original paper for figures.