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Abstract

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MATHEMATICS

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ON COMPARING BASES IN THE REALIZATION OF FUNCTIONS OF THE ALGEBRA OF LOGIC BY FORMULAS

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In the present work we study the realization of functions of the algebra of logic by formulas in finite bases. As is known, from the point of view of the complexity of the realization of almost all functions, bases are in a certain sense equivalent (1). However, when individual functions are realized, differences between bases are revealed. In this work an order and equivalence relation is introduced on the set of bases, the existence of nonequivalent bases is proved, the existence in a certain sense of the worst basis is proved, namely the basis $\{\&, \vee, ^{-}\}$, and a necessary and sufficient condition is established for equivalence to this basis.

§ 1. Let $B = \{\varphi_1, \dots, \varphi_n\}$ be a complete system of functions of the algebra of logic. Expressions of the form $\varphi_i(x_1, \dots, x_{i_k})$ $(1 \leq i \leq n;\ k_i$ is the number of essential variables of the function $\varphi_i)$ will be called **basis formulas**. We shall consider formulas that are superpositions of basis formulas (“formulas in the basis B”). Two formulas realizing one and the same function will be called **equivalent**; at the same time we shall not distinguish functions that differ by inessential variables. Let $L(F)$ be the number of variable symbols in the formula F, and $L_B(f) = \min L(F)$ (the minimum is taken over all formulas in the basis B that realize the function f)*.

Definition. We shall say that a basis B_1 **precedes** a basis B_2 (notation $B_1 \preccurlyeq B_2$) if there exists a constant M (depending only on B_1 and B_2) such that for any function f the inequality $L_{B_1}(f) \leq M L_{B_2}(f)$ holds. We shall say that the bases B_1 and B_2 are **equivalent** (notation $B_1 \sim B_2$) if: a) $B_1 \preccurlyeq B_2$ and b) $B_2 \preccurlyeq B_1$. If condition a) is fulfilled, while b) is not, then we shall say that the basis B_1 **strictly precedes** the basis B_2.

We shall call a formula F **repetition-free** if every essential variable of the function realized by this formula has exactly one occurrence in F.

Lemma 1. *If all functions of the basis B_1 are expressed by repetition-free formulas in the basis B_2, then $B_2 \preccurlyeq B_1$**.

Theorem 1. *If all functions of the basis B_1 are expressed by repetition-free formulas in the basis B_2, and conversely, then $B_1 \sim B_2$.*

Lemma 2. *The constants $0, 1$, negation, and nonlinear functions of two variables are realized by repetition-free formulas in any basis.*

The proof almost literally coincides with the proof of Lemma 1 in (1).

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[*] All results of this work remain valid if positive weights are assigned to the functions of the basis and $L(F)$ is defined as the sum of the weights of the basis functions occurring in F (taking their multiplicities into account).

[**] A formula constructed from repetition-free formulas has the same structure as a formula constructed from basis formulas (nothing need be substituted for inessential variables).

Denote by the symbol B_0 the basis $\{\&, \vee, {}^{-}\}$. From Lemmas 1 and 2 it follows that

Theorem 2*. *Every basis B satisfies the relation $B \preccurlyeq B_0$.*

Theorem 3. *There exist nonequivalent bases **.*

Examples of nonequivalent bases are B_0 and $B_1 = \{\&, +(\operatorname{mod} 2), 1\}$. For the function $f_n = x_1 + \dots + x_n (\operatorname{mod} 2)$ we have, on the one hand, $L_{B_1}(f_n) = n$, and, on the other hand, $L_{B_0}(f_n) > Cn^{3/2}$ (3). Other examples of nonequivalent bases are easily constructed on the basis of Theorem 4 of this paper.

§ 2. Functions obtained from a function f of the algebra of logic as a result of substituting constants in place of some variables, as well as the function f itself, will be called **derivatives** of the function f. We shall say that an essential variable of a function has a **distinguished value** τ, if ${}^{\tau}f^{x_i}$ (3) *** does not depend on some essential variable x_j $(x_j \neq x_i)$ of the function f (i.e. ${}^{\tau 0}f^{x_i x_j} = {}^{\tau 1}f^{x_i x_j}$); in this case we shall also say that x_i **absorbs** x_j in f (with distinguished value τ). Obviously,

$$\text{if } x_i \text{ absorbs } x_j, \text{ and } x_j \text{ absorbs } x_k, \text{ then } x_i \text{ absorbs } x_k. \tag{1}$$

A variable that has no distinguished value will be called **marked**. Let Φ be the set of functions all derivatives of which have no marked variables ****. Obviously, among the functions of two variables the set Φ contains all nonlinear functions and only them. It is also obvious that together with every function f the set Φ contains any function obtained from f by replacing some variables by their negations.

Lemma 3. *Every function $f(x_1, \dots, x_n)$ from Φ, $n > 2$, is representable in the form $f_1(\varphi(x_{i_1}, x_{i_2}), x_{i_3}, \dots, x_{i_n})$, where $\varphi, f_1 \in \Phi$.*

Proof. Each variable x_i of the function f absorbs some other variable x_j. From (1) it follows that there is a pair of variables x_{i_1} and x_{i_2} such that x_{i_1} absorbs x_{i_2}

(with some distinguished value τ_1) and x_{i_2} absorbs x_{i_1} (with some distinguished value τ_2) *****. Therefore

$$ {}^{\tau_1\bar{\tau}_2}f^{x_{i_1}x_{i_2}} = {}^{\tau_1\tau_2}f^{x_{i_1}x_{i_2}} = {}^{\bar{\tau}_1\tau_2}f^{x_{i_1}x_{i_2}} \qquad (2) $$

and, upon substituting constants in place of the variables x_{i_1} and x_{i_2} in the function f, no more than two distinct derivatives are obtained. Consequently, the set $\{x_{i_1}, x_{i_2}\}$ of variables is separable ((4), p. 192). The first part of the lemma is proved.

Now let $f(x_1, \dots, x_n) = f_1(\varphi(x_{i_1}, x_{i_2}), x_{i_3}, \dots, x_{i_n})$. From (2) it follows that φ is a nonlinear function and therefore belongs to Φ. The function $f_1(x_{i_2}, x_{i_3}, \dots, \dots, x_{i_n})$ is obtained from f by substituting some constant in place of x_{i_1} and, possibly, replacing x_{i_2} by $\bar{x}_{i_2}$; therefore $f_1 \in \Phi$.

Corollary. Any function $f(x_1, \dots, x_n)$ from Φ, $n \geq 2$, can be represented as a superposition of nonlinear functions of two variables, in which all variable symbols are pairwise distinct (in the form of a repetition-free superposition).

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* Theorems 1, 2 and Lemmas 1, 2 were proved by O. B. Lupanov.
** For circuits of functional elements all finite bases turn out to be equivalent (2).
*** In this paragraph some notions and facts from (3) are used.
**** There exist functions having no marked variables, but some derivatives of which do have marked variables, for example, $x_1x_2x_3x_4 \vee \bar{x}_1\bar{x}_2x_5x_6$.
***** Consider the sequence $x_{i_1}, \dots, x_{i_k}, \dots$, where x_{i_s} absorbs $x_{i_{s+1}}$. In this sequence there is a pair of identical variables, say, x_{i_p} and x_{i_q}. They are not adjacent. Therefore x_{i_p} absorbs $x_{i_{p+1}}$ and $x_{i_{p+1}}$ absorbs x_{i_p}.

Lemma 4. A function f is realizable without repetition in the basis B_0 if and only if $f \in \Phi^*$.

Theorem 4. If in the basis B there is a function φ such that some derivative ψ of it (in particular, $\psi = \varphi$) has a distinguished variable and depends essentially on more than one variable (i.e., $\varphi \notin \Phi$), then B strictly precedes the basis B_0.

Proof. Let $\psi(x_1, \dots, x_n)$ be a function with distinguished variable x_1, which is a derivative of the basis function φ and depends essentially on n variables, $n \geq 2$. Consider the sequence of functions

$$ \psi_1(x_1, \dots, x_n) = \psi(x_1, \dots, x_n), $$

$$ \psi_i = \psi_{i-1}(\psi^1, \dots, \psi^{n^{i-1}}), \qquad i = 2, 3, \dots, $$

where ψ^j is the function obtained from $\psi(x_1, \dots, x_n)$ by replacing the variables $x_1, \dots, x_n$, respectively, by $x_1^j, \dots, x_n^j$ (the variables $x_{i_1}^{j_1}, x_{i_2}^{j_2}$, corresponding to different pairs (i_1, j_1), (i_2, j_2), are distinct). We shall show that

$$\frac{L_{B_0}(\psi_i)}{L_B(\psi_i)} \to \infty \qquad (i \to \infty).$$

Thus, in view of Theorem 2, this theorem will be proved. The set $\{x_1^j, \dots, x_n^j\}$ of variables will be called a subset; a set of n^{i-1} variables (of the function ψ_i) containing one variable from each subset will be called a special set. Let F be a formula in the basis B_0 realizing the function ψ_i and such that $L(F) = L_{B_0}(\psi_i)$. By Lemma 2, from (3) it may be assumed that F is a normal formula. We describe a certain partition of all variables into n special sets $r_1, \dots, r_n$. The set r_1 will be constructed in a special way, while the others are chosen arbitrarily subject to only one condition: each variable from each subset belongs to one of the special sets $r_1, \dots, r_n$. The special set r_1 is the union of two disjoint sets A and C of variables. We construct A, and simultaneously an auxiliary sequence $B = \{b_1, \dots, b_\nu\}$ of variables, $\nu = \left[\frac{n^{i-1}}{2}\right]$. Put x_1^1 into A. Let τ_1 be a defining value (3) of some occurrence of the variable x_1^1 in F; let, when τ_1 is substituted in F in place of x_1^1, some occurrence of the variable $x_{i_1}^{j_1}$, different from x_1^1, become inessential, and let F_1 be the normal formula corresponding to ${}^{\tau_1}F^{x_1^1}$ (3). Put $b_1 = x_{i_1}^{j_1}$. Then put into A the variable $x_1^{j_2}$, different from $x_{i_1}^{j_1}$ (and from x_1^1). Let τ_2 be a defining value of some occurrence of the variable $x_1^{j_2}$ in F_1; let, when τ_2 is substituted in F_1 in place of $x_1^{j_2}$, some occurrence of the variable $x_{i_2}^{j_2}$, different from $x_1^{j_2}$ (and, obviously, from x_1^1; note that $x_{i_2}^{j_2}$ may coincide with $x_{i_1}^{j_1}$), become inessential, and let F_2 be the normal formula corresponding to ${}^{\tau_2}F_1^{x_1^{j_2}}$. Put $b_2 = x_{i_2}^{j_2}$, etc. This process will continue until A contains ν variables from ν subsets and B has ν terms; B will contain no more than ν distinct variables. Let $N_1, \dots, N_\mu$ $(\mu + \nu = n^{i-1})$ be those subsets whose variables did not enter A, and let p_k be the number of occurrences of variables from N_k in B $(1 \le k \le \mu)$. Put into C, from each subset N_k, that variable which occurs the least number of times in B. This number of occurrences does not exceed $\frac{p_k}{n}$. Therefore in B there will be no more than

$$\frac{1}{n}(p_1 + \dots + p_\mu) \le \frac{\nu}{n}$$

occurrences

* If f is not a constant and is realizable without repetition in the basis B_0, then f is also realized by a repeti

of occurrences of variables from C and at least $\nu - \frac{\nu}{n}$ occurrences of variables not contained in C (nor in A). Upon substituting in F, in place of the variables $x_1^1, x_1^2, \dots, x_1^\nu$ from A (i.e., from r_1), respectively the constants $\tau_1, \dots, \tau_\nu$, one obtains a formula F' realizing a function essentially depending on all the variables from $C, r_2, \dots, r_n$ and containing at least $\nu - \frac{\nu}{n}$ inessential occurrences of variables from $r_2, \dots, r_n$. Let q_s be the number of such inessential occurrences of variables from r_s, $2 \le s \le n$ (therefore

$$q_2 + \dots + q_n \ge \left(1 - \frac{1}{n}\right)\nu > \left(1 - \frac{1}{n}\right)\left(\frac{n^{i-1}}{2} - 1\right) > n^{i-4} \quad \text{for } i \ge 3$$

).

Then the number $L^s(F)$ of occurrences of variables from r_s in the formula F satisfies the relation

$$L^s(F) \ge L_{B_0}(\psi_{i-1}) + q_s, \qquad 2 \le s \le n,$$

since from the formula F', by a suitable substitution of constants for all variables not occurring in r_s, one can obtain a formula expressing a function of the same type as ψ_{i-1}.* Therefore

$$L(F) = \sum_{s=1}^{n} L^s(F) \ge nL_{B_0}(\psi_{i-1}) + \sum_{s=2}^{n} q_s \ge nL_{B_0}(\psi_{i-1}) + n^{i-4} \quad (i \ge 3)$$

and

$$L_{B_0}(\psi_i) \ge nL_{B_0}(\psi_{i-1}) + n^{i-4} \quad (i \ge 3). \tag{3}$$

Starting from (3), it is easy to show by induction on i that, for $i \ge 3$,

$$L_{B_0}(\psi_i) \ge (i-2)n^{i-4}.$$

On the other hand, since ψ is realized in the basis B without repetitions, we have

$$L_B(\psi_i) \le Cn^i,$$

where C is some constant. Therefore

$$\frac{L_{B_0}(\psi_i)}{L_B(\psi_i)} \to \infty \qquad (i \to \infty).$$

The theorem is proved.

Theorem 5. *The basis B is equivalent to B_0 if and only if all functions from B are realized without repetitions in B_0.*

This theorem follows immediately from Theorem 4 and Lemma 4.

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* That is, obtained from ψ_{i-1} by replacing its variables with variables from r_s or their negations.

Note: Figure translations are in progress. See original paper for figures.

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