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Abstract

Full Text

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IRREDUCIBLE UNITARY REPRESENTATIONS OF THE GROUP OF UNIMODULAR MATRICES OF THE SECOND ORDER WITH ELEMENTS FROM A LOCALLY COMPACT FIELD

1. We consider the group $G = SL(2, K)$ of second-order matrices with determinant 1 whose elements belong to an arbitrary locally compact nondiscrete field K . In this paper irreducible unitary representations of the group G are constructed and the traces of these representations are computed.* We first formulate several known facts about locally compact nondiscrete fields.

A. Every connected locally compact field is isomorphic to the field R of real numbers or to the field C of complex numbers. Every disconnected locally compact nondiscrete field of characteristic 0 is isomorphic to a finite extension of the field Q_p of p -adic numbers (p is some prime number). Every disconnected locally compact nondiscrete field of characteristic $p \neq 0$ is isomorphic to a finite extension of the field $F_p(t)$ of formal power series $a_{-s}t^{-s} + \dots + a_0 + a_1t + \dots$ with coefficients in a finite field of order p (see (3)).

B. Let K^+ be the additive group of the field K ; K^* its multiplicative group; dx the Haar measure on K^+ . Define a function $|a|$ on K by the formula $d(xa) = |a|dx$. It is evident that the function $|a|$ has the following properties: 1) $|0| = 0$ and $|a| > 0$ for $a \neq 0$; 2) $|ab| = |a| \cdot |b|$ for any a, b in K . The function $|a|$ is a norm in K , except in the case when $K = C$. In the case $K = C$, the function $|a|$ is the square of the norm. Clearly $d^*x = |x|^{-1}dx$ is a Haar measure on K^* . Hence it is easy to obtain that, under the fractional-linear transformation

$$x' = xg = \frac{\alpha x + \gamma}{\beta x + \delta},$$

the measure is transformed according to the formula

$$dx' = |\beta x + \delta|^{-2} |\det g| dx.$$

C. Let the field K be disconnected. Then its multiplicative group K^* is isomorphic to the direct product $Z \times Z_q \times A$ of an infinite cyclic group Z , a finite cyclic group Z_q of order $q = p^n - 1$ (p is a prime number), and the group A of elements

x of K for which $|x - 1| < 1$. It is easy to show that for $p \neq 2$ every element of A is representable as the square of some other element of A . Hence it follows that in the case $p \neq 2$ the field K has exactly 3 quadratic extensions: $K(\sqrt{\tau_0})$, $K(\sqrt{\tau_q})$, and $K(\sqrt{\tau_0\tau_q})$, where τ_0, τ_q are generating elements of the subgroups Z and Z_q .

D. We shall call each quadratic extension $K(\sqrt{\tau})$ of the field K a “complex” plane, and its elements $z = x + \sqrt{\tau}y$ “complex” numbers. By the symbol \bar{z} we shall denote the complex number conjugate to z . Elements of K^* representable in the form $z\bar{z} \equiv x^2 - \tau y^2$, $x, y \in K$, will be called positive (with respect to the given extension), and the remaining elements of K^* will be called negative. Introduce on K^* the function $\text{sign } x$: $\text{sign } x = 1$ if x is positive; $\text{sign } x = -1$ if x is negative. It is easy to show that the set of negative numbers is nonempty and that the function $\text{sign } x$ is a character on K^* , i.e. $\text{sign}(xy) = \text{sign } x \cdot \text{sign } y$ for any x, y in K^* .

E. The additive group K^+ of a locally compact nondiscrete field is dual to itself. We shall denote by $\chi(x)$ the characters on K^+ ($\chi(x+y) = \chi(x)\chi(y)$, $|\chi(x)| = 1$). Let $\chi \neq 1$ be a fixed character on K^+ . Then any character on K^+ is uniquely determined by a number $u \in K^+$ and has the form $\chi_u(x) = \chi(ux)$.

* Representations of the group of matrices of order 2 with elements from a finite field were considered in (2); the case of the complex and real fields was considered in (1,5).

2. Let us give a description of the principal continuous series of irreducible unitary representations of the group G . The construction of these representations for the case of the field K of complex numbers is well known ⁽¹⁾; this construction carries over automatically also to the case of any locally compact nondiscrete field K . A representation of the continuous series is specified by a character $\pi(x)$ on the group K^* (i.e., by a continuous complex-valued function satisfying the conditions $\pi(xy) = \pi(x)\pi(y)$, $|\pi(x)| = 1$); it is constructed in the space of functions $f(x)$ on K for which

$$\|f\|^2 = \int |f(x)|^2 dx < \infty.$$

The representation operator $T_\pi(g)$, corresponding to the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, has the form

$$T_\pi(g)f(x) = f\left(\frac{\beta + \delta x}{\alpha + \gamma x}\right) \pi(\alpha + \gamma x)|\alpha + \gamma x|^{-1}.$$

It can be shown that the representations $T_\pi(g)$ are irreducible (see ^(1,5,4)). *Two representations of the continuous series, corresponding to the characters π_1, π_2 , are equivalent if and only if either $\pi_1 = \pi_2$, or $\pi_1 = \pi_2^{-1}$.*

We obtain another realization of the representations of the continuous series if

we pass from the functions $f(x)$ to their Fourier transforms

$$\varphi(u) = \int f(x)\chi(-ux) dx,$$

where $\chi \neq 1$ is a fixed character of K^+ . In this realization the operator $T_\pi(g)$ has the form

$$T_\pi(g)\varphi(u) = \int K(g; u, v)\varphi(v) dv,$$

where the kernel $K(g; u, v)$ is a generalized function defined by the formula

$$K(g; u, v) = |\gamma|^{-1}\chi\left(\frac{u\alpha + v\delta}{\gamma}\right) \int \chi\left(-\frac{1}{\gamma}(ux + vx^{-1})\right) \pi(x) d^*x, \quad (1)$$

if $\gamma \neq 0$;

$$K(g; u, v) = |\alpha| \cdot \pi(\alpha)\chi(\alpha\beta u) \delta(v - \alpha^2 u),$$

if $\gamma = 0$ ($\delta(u)$ is the delta-function on K^+). The integral appearing in formula (1) will be called a Bessel function of the first kind (cf. (2)).

3. With each quadratic extension $K(\sqrt{\tau})$ of the field K we associate a discrete series of irreducible unitary representations of the group

$$G = SL(2, K).$$

The formulas for the operators of the representations of this series are obtained from (1) by “analytic continuation.” Namely, in (1) we replace the character π on K^* by an arbitrary character on the “complex” plane $K^*(\sqrt{\tau})$, and the integration in (1) will be carried out not along the “real axis” K , but along a contour in $K(\sqrt{\tau}) \setminus K$, along which the expression

$$-\frac{1}{\gamma}(uz + vz^{-1})$$

belongs to the original field K . It is easy to see that this contour consists of those z for which $z\bar{z} = vu^{-1}$. (We note that if vu^{-1} is negative, then such a contour does not exist, and in this case we must put $K(g; u, v) = 0$.)

To each multiplicative character $\pi(z)$ on $K(\sqrt{\tau})$ there corresponds a pair of irreducible unitary representations $T_\pi^+(g)$ and $T_\pi^-(g)$ of the group G . The representation $T_\pi^+(g)$ acts in the space of functions $\varphi(u)$ that are equal to zero when $\text{sign } u = -1$ and are such that

$$\|\varphi\|^2 = \int |\varphi(u)|^2 du < \infty.$$

Analogously, the representation $T_\pi^-(g)$ acts in the space of functions $\varphi(u)$ that are equal to zero when $\text{sign } u = 1$. The operators $T_\pi^+(g)$, $T_\pi^-(g)$ have the form

$$T_\pi^+(g)\varphi(u) = \int K(g; u, v)\varphi(v) dv, \quad T_\pi^-(g)\varphi(u) = \int K(g; u, v)\varphi(v) dv,$$

where

$$K(g; u, v) = c_\chi \operatorname{sign} u \cdot \operatorname{sign} \gamma \cdot |\gamma|^{-1} \chi\left(\frac{u\alpha + v\delta}{\gamma}\right) \int_{z\bar{z}=vu^{-1}} \chi\left(-\frac{1}{\gamma}(z + z^{-1})\right) \pi(z) d^*z, \quad (2)$$

if $\gamma \neq 0$, $\operatorname{sign}(vu^{-1}) = 1$;

$$K(g; u, v) = \operatorname{sign} \alpha \cdot |\alpha| \cdot \pi(\alpha) \cdot \chi(\alpha\beta u) \delta(v - \alpha^2 u),$$

if $\gamma = 0$; $K(g; u, v) = 0$, if $\operatorname{sign}(vu^{-1}) = -1$.

Here $\bar{c}_\chi^{-1} = \int \chi(z\bar{z}) dz$ (the integral is taken over the complex plane $K(\sqrt{\bar{\tau}})$). It is not difficult to show that $c_\chi = \pm 1$ or $c_\chi = \pm i$. The integration in (2) is with respect to the measure d^*z on the circle $z\bar{z} = vu^{-1}$, determined by the conditions: $\int d^*z = 1$, $d^*(zz_0) = d^*z$ for any z_0 such that $z_0\bar{z}_0 = 1$. Since the circle $z\bar{z} = vu^{-1}$ is compact, the integral certainly converges. The integral

$$- \int_{z\bar{z}=vu^{-1}} \chi\left(-\frac{1}{\gamma}(z + z^{-1})\right) \pi(z) d^*z$$

is naturally called a Bessel function of the second kind (in the case of the field of real numbers this integral is equal to $J_n(\sqrt{uv}/\gamma)$, where n is the number of the character π , and J_n is the Bessel function of index n).

The representations $T_{\pi_1}^+(g)$ and $T_{\pi_2}^+(g)$ (respectively, $T_{\pi_1}^-(g)$ and $T_{\pi_2}^-(g)$) are equivalent if and only if on the circle $z\bar{z} = 1$ either $\pi_1(z) = \pi_2(z)$, or $\pi_1(z) = \pi_2^{-1}(z)$. Thus, the set of mutually inequivalent representations $T_\pi^+(g)$ is isomorphic to the set of characters on the circle $z\bar{z} = 1$, in which the characters π and π^{-1} are identified; consequently, this set of representations is discrete. The representations $T_{\pi_1}^+(g)$ and $T_{\pi_2}^-(g)$ are not equivalent to one another. Two representations belonging to different series are also not equivalent to one another.

The representations of the continuous and discrete series exhaust all irreducible unitary representations of the group G contained in the regular representation.

4. Let $T(g)$ be any one of the representations of the group G considered above. Then the trace $\operatorname{Tr} T(g)$ of the operator $T(g)$ is defined as a generalized function in a suitable space of test functions on G . Below are given formulas for $\operatorname{Tr} T(g)$ on the set of regular elements of the group G (that is, on the set of matrices g with distinct eigenvalues).

Let $T_\pi(g)$ be a representation of the continuous series corresponding to the character $\pi(x)$ on K . If the eigenvalues λ, λ^{-1} of the matrix g belong to K , then

$$\operatorname{Tr} T_\pi(g) = \frac{\pi(\lambda) + \pi(\lambda^{-1})}{|\lambda - \lambda^{-1}|};$$

otherwise $\operatorname{Tr} T_\pi(g) = 0$. Let now $T_\pi^+(g), T_\pi^-(g)$ be the representations of the discrete series described in Sec. 3; λ, λ^{-1} are the eigenvalues of the matrix g . Then

$$\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g) = c_\chi \operatorname{sign} \gamma \cdot \frac{\pi(\lambda) + \pi(\lambda^{-1})}{|\lambda - \lambda^{-1}|},$$

if λ and λ^{-1} belong to $K(\sqrt{\tau})$ and do not belong to K ; $\text{Tr } T_{\pi}^{+}(g) - \text{Tr } T_{\pi}^{-}(g) = 0$ in the remaining cases. Further, we have

$$\text{Tr } T_{\pi}^{+}(g) + \text{Tr } T_{\pi}^{-}(g) = 2 \int_{z\bar{z}=1} \frac{\text{sign}(\alpha + \delta - z - \bar{z})}{|\alpha + \delta - z - \bar{z}|} \pi(z) d^*z.$$

This integral is easily evaluated when K is the field of real numbers. In the case of a disconnected field K it can be shown that if the eigenvalues of the matrix g belong either to K , or to a quadratic extension of the field K different from $K(\sqrt{\tau})$, then $\text{Tr } T_{\pi}^{+}(g) = \text{Tr } T_{\pi}^{-}(g) = 0$ for all π , except, perhaps, for a finite number of them (depending on g).

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Note: Figure translations are in progress. See original paper for figures.

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