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Abstract

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MATHEMATICS

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SOME CLASSES OF VECTOR CHAINS AND THEIR APPLICATIONS TO THE THEORY OF PARTIALLY ORDERED SPACES

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§ 1 is introductory in character; in § 2 the question of the isomorphism of certain (completely) ordered vector spaces is considered, and l-ideals in such spaces are studied; in § 3 the results obtained are applied to the theory of partially ordered spaces, in particular to the question of the uniqueness of the extension of a regular operator from an Archimedean K-linear X to its K-completion and to a criterion for the disjointness of regular operators in X.

§ 1. F. Hausdorff, in his well-known monograph [10], introduced and investigated the concept of an η_α-set*.

A nonempty ordered set H^{**} is called an η_α-set if the following conditions are satisfied in it.

I. If $A, B \subset H$ are nonempty, $\overline{\overline{A}}, \overline{\overline{B}} < \aleph_\alpha$, and $A < B$ (i.e., $a < b$ for any $a \in A$, $b \in B$), then there exists an element $h \in H$ such that $A < h < B$.

II. If K is a confinal (or coinitial) subset in H, then

$$\overline{\overline{K}} \geq \aleph_\alpha.$$

F. Hausdorff showed that, in the case of singular $\aleph_\alpha$, every η_α-set is also an $\eta_{\alpha+1}$-set. In view of this, in what follows we shall consider only the case of regular $\aleph_\alpha$, without mentioning this separately.

The least cardinality of an $\eta_{\alpha+1}$-set is $2^{\aleph_\alpha}$. We shall assume the generalized continuum hypothesis to be true, and then there exist (at any rate for non-limit α) η_α-sets of cardinality $\aleph_\alpha$. F. Hausdorff proved that any two η_α-sets of cardinality $\aleph_\alpha$ are isomorphic (as ordered sets).

Subsequently, some authors considered η_α-sets endowed with algebraic operations compatible with the order; for example, η_α-rings (η_α-sets that are associative commutative rings, with $x, y > 0$ implying $x + y > 0$, $xy > 0$), and η_α-fields.

In particular, the question of the isomorphism of the corresponding sets was considered.

It is known that there exist nonisomorphic η_α-rings of cardinality $\aleph_\alpha$ (see, for example, [(6)]). P. Erdős, L. Gillman, and M. Henriksen [(12)] proved the isomorphism of η_α-fields ($\alpha \geq 1$) of cardinality $\aleph_\alpha$ in the class of real-closed fields (for the definition and properties of real-closed fields see, for example, [(1)]).

§ 2. We shall consider vector η_α-spaces.

Definition. An η_α-linear is any η_α-set $\mathcal{L}$ that is a vector (linear) ordered space –

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[*] This concept did not enter either the second edition of the monograph or its Russian translation. For this concept see also [(9)].

[**] By an ordered set we shall throughout understand a completely ordered set, i.e., one in which any two distinct elements h_1, h_2 are comparable: $h_1 < h_2$ or $h_2 < h_1$.

over the field of real numbers (the natural compatibility of the linear operations and the order is assumed: if $x, y \in \mathcal{L}$, $x, y > 0$, and the real number $\lambda > 0$, then $x + y > 0$ and $\lambda x > 0$).

For $\alpha \geqslant 1$, if there exists an η_α-set of cardinality $\aleph_\alpha$, then there also exists an η_α-lineal of cardinality $\aleph_\alpha$.

Theorem 1. *For a given α, all η_α-lineals of cardinality $\aleph_\alpha$ are isomorphic (as vector ordered spaces).*

The most difficult case to prove is the case $\alpha = 1$.

Corollary. *Let $\mathcal{L}$ be an η_α-lineal of cardinality $\aleph_\alpha$, $y \in \mathcal{L}$, $y > 0$. Then the l-ideal (normal sublineal) of $\mathcal{L}$*

$$\mathcal{L}_y = \{x \in \mathcal{L} : \ x \ll y\}$$

is isomorphic to $\mathcal{L}$ itself.

We introduce the notion of an η_α^δ-lineal ($\delta = -1$ or is an ordinal number $\delta \leqslant \alpha$ with regular $\aleph_\delta$). In a vector ordered space H, besides conditions I–II the following conditions may also hold.

III. In H there exists a **strong unit**, i.e., an element $e > 0$ such that for every $x \in H$

$$-\lambda e < x < \lambda e$$

for some real λ, depending on x.

IV. The least cardinality of a cofinal subset in H is $\aleph_\delta$.

Definition. A vector ordered space of cardinality $\aleph_\alpha$ is called an η_α^{-1}**-lineal** if conditions I and III hold in it; it is called an η_α^δ**-lineal** (for $0 \leqslant \delta \leqslant \alpha$) if conditions I and IV hold and III does not hold (the failure of III need be stipulated only for $\delta = 0$, since III entails the existence of a countable cofinal subset $\{e, 2e, 3e, ...\}$).

Obviously, the notion of an η_α^α-lineal coincides with the notion of an η_α-lineal of cardinality $\aleph_\alpha$. The following theorem holds, which for $\delta = \alpha$ turns into Theorem 1.

Theorem 2. *For given $\delta \leqslant \alpha$, all η_α^δ-lineals are isomorphic (as vector ordered spaces).*

We next give a description of all nonempty l-ideals (normal sublineals) of an η_α-lineal of cardinality $\aleph_\alpha$.

Theorem 3. *Let $\mathcal{L}$ be an η_α-lineal of cardinality $\aleph_\alpha$. Every nonempty l-ideal of it is an η_α^δ-lineal for some $\delta \leqslant \alpha$. Conversely, if $\delta = -1$ or $0 \leqslant \delta \leqslant \alpha$ and $\aleph_\delta$ is regular, then in $\mathcal{L}$ there exists a nonempty l-ideal which is an η_α^δ-lineal.*

An l-ideal of $\mathcal{L}$ which is an η_α^{-1}-lineal is any l-ideal of the form

$$\mathcal{L}^y = \{x \in \mathcal{L} : \ -\lambda y < x < \lambda y \text{ for a real } \lambda = \lambda(x)\}$$

$(y \in \mathcal{L},\ y > 0)$. For $\delta \neq -1$, an l-ideal $\mathcal{L}$ which is an η_α^δ-lineal can be obtained as follows. Let K be a subset of $\mathcal{L}$ that is fully ordered by type Ω_δ with respect to the relation $\ll$,

$$K = \{y_\beta\} \qquad (\beta < \Omega_\delta)$$

(if $\beta' < \beta''$, then $y_{\beta'} \ll y_{\beta''}$). Then the desired l-ideal will be

$$\mathcal{L}_K = \{x \in \mathcal{L} : \ x \ll y_\beta \text{ for some } \beta = \beta(x)\}.$$

In the case $\delta = \alpha$, the indicated construction gives a description of all l-ideals of $\mathcal{L}$ isomorphic to $\mathcal{L}$. Among them there will be l-ideals not representable in the form $\mathcal{L}_y$ (see the corollary to Theorem 1). This fact follows from the following theorem.

Theorem 4. *Let $\mathcal{L}$ be an η_α-lineal of cardinality $\aleph_\alpha$. Then the intersection of any system, of cardinality less than $\aleph_\alpha$, of its l-ideals, each of which is isomorphic to $\mathcal{L}$, is itself isomorphic to $\mathcal{L}$.*

§ 3. Earlier the notions of an η_1-ring and an η_1-field were applied to the study of partially ordered rings of continuous functions ([12,7,8]) and of Φ-algebras (partially ordered Archimedean rings with a weak unit which is the multiplicative identity) ([11]). Here the notion of an η_1-lineal and certain results of § 2 are applied to the theory of (linear) partially ordered spaces. Throughout § 3 we shall assume the continuum hypothesis to be true. Unless otherwise stated, we shall use the terminology of ([5]).

Let us recall some notions. An l-ideal I of a K-lineal X is called **simple** if from $x \wedge y \in I,\ x \in I$ it follows that $y \in I$. A simple l-ideal is called **minimal** if it contains no smaller simple l-ideal.

A **component** of a K-lineal X is any set that is the disjoint complement of some $M \subset X$. The smallest component containing a given element is called the **component generated by the element**, or the **principal component**. If for every $x \in X$ and every component X' there exists a projection of x into X' (for $x > 0$ this means the existence of $\sup\{x' \in X' : 0 \le x' \le x\}$), then X is called a **K-lineal with projections**.

Theorem 5. *Let X be an Archimedean K-lineal, the cardinality of every principal component of which is equal to the cardinality of the continuum c (in particular, an Archimedean K-lineal of cardinality c). Let P be its minimal simple l-ideal. Then the quotient X/P is either isomorphic to the real line (i.e. the l-ideal P is material), or is an η_1^δ-lineal for $\delta = -1, 0,$ or 1.*

Corollary. *There exist nonisomorphic K-spaces X_1 and X_2, containing minimal simple l-ideals P_1 and P_2, respectively, such that P_1 and P_2 are isomorphic and the quotients X_1/P_1 and X_2/P_2 are also isomorphic.*

Definition. A K-space is called a space of **type** s^* if every bounded set of its pairwise disjoint elements is finite. A space of type s^* is always discrete.

Theorem 6. *Let $\hat{X}$ be a K-space satisfying at least one of the following conditions:*

1) *The discrete part of $\hat{X}$ is not a space of type s^*.*

2) *The continuous part of $\hat{X}$ has at least one principal component of cardinality c.*

Then there exists a K-lineal X, not a K-lineal with projections, whose K-completion is $\hat{X}$.

Remark 1. In the formulation of the theorem one may additionally require that in $\hat{X}$ there exist a projection of every element into every principal component.

Remark 2. It is easy to see that the restriction imposed in the theorem on the discrete part of $\hat{X}$ is essential. The restriction on the continuous part, however, is not. The method used by the author for proving the theorem does not yet make it possible to get rid of this restriction.

In ([4]) the following assertions were considered; they can be carried over to an Archimedean K-lineal X ($\hat{X}$ is the K-completion of X).

A. For some K-space Y, the spaces $(X \to Y)_r$ and $(\hat{X} \to Y)_r$ of regular operators from X to Y and from $\hat{X}$ to Y coincide (i.e. every regular operator from X to Y admits a unique extension to all of $\hat{X}$).

B. For every Y, $(X \to Y)_r$ and $(\hat{X} \to Y)_r$ coincide.

V. For some Y the following disjointness criterion for regular operators is satisfied. If $U_1, U_2 \in (X \to Y)_r$, then for their disjointness it is sufficient that there exist two mutually complementary components X_1 and X_2 such that $U_1(X_1) = U_2(X_2) = \{0\}$.

G. For every Y in $(X \to Y)_r$ the disjointness criterion is satisfied.

D. X is a K-lineal with projections.

Definition. We shall say that a K-space $\hat{X}$ has **property (E)** if, for every Archimedean K-lineal X whose K-completion is $\hat{X}$, the assertions A, B, D can be fulfilled only simultaneously; and we shall say that $\hat{X}$ has **property (P)** if the same holds for V, G, D.

It was previously established by the author that a K-space of bounded elements (i.e., a K-space having a strong unit) has property (E) ([3], Theorems 2 and 3) and property (P) ([2]). In ([4]) (Theorem 5) this result was generalized to the case of an arbitrary K-space that is an MK-lineal (i.e., a K-lineal in which every l-ideal can be embedded in a maximal one). Now, considering a very broad class of spaces, we indicate in this class all K-spaces possessing properties (E) and (P).

Definition. A finite or infinite cardinal number $m = m(\hat{X})$, equal to the number of minimal prime l-ideals of the given K-space, each of which cannot be embedded in a maximal l-ideal, is called the **MK-defect** of $\hat{X}$.

The MK-defect of a K-space that is an MK-lineal (in particular, of a K-space of bounded elements) is equal to 0.

Theorem 7. *Let $\hat{X}$ be a K-space in which every principal component has cardinality c, and let m be its MK-defect. Then, if $m \leqslant 1$, $\hat{X}$ has properties (E) and (P); if $m \geqslant 3$, then $\hat{X}$ does not have these properties; finally, there exist K-spaces for which $m = 2$, both possessing properties (E) and (P) and not possessing them.*

It would be interesting, if possible, to obtain the results of this paragraph without assuming the validity of the continuum hypothesis.

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Note added in proof. After the present paper had been submitted for publication, the author learned of the works of N. Alling ([13]). N. Alling proved that if, for some $\alpha > 0$, there exists an η_α-set of cardinality $\aleph_\alpha$, then there exist nonisomorphic η_α-fields of cardinality $\aleph_\alpha$ and nonisomorphic η_α-groups of cardinality $\aleph_\alpha$; he proved isomorphism of η_α-groups of cardinality $\aleph_\alpha$ in the class of commutative groups with division.

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Note: Figure translations are in progress. See original paper for figures.