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Abstract

Full Text

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On the Green' s Function for the Diffraction Problem on a Polyhedral Angle

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1. Let us consider, as $k \to \infty$, the asymptotics of the Green' s function $G(\mathbf{x}, \mathbf{a}, k)$ for the stationary diffraction problem on a polyhedron S, i.e., the solution of the equation $(\Delta_x + k^2)G = \delta(\mathbf{x} - \mathbf{a})$ with boundary conditions prescribed on S. Here $\mathbf{x} = x_1, x_2, x_3$ is the position of the observation point; $\mathbf{a} = a_1, a_2, a_3$ is the position of the source. We shall assume that ideal boundary conditions are prescribed on the polyhedron S,

$$G\big|_S = 0 \quad \text{or} \quad \left.\frac{\partial G}{\partial n}\right|_S = 0.$$

As is known, these asymptotics are determined by the singularities (behavior in a neighborhood of wave fronts) of the Green' s function for the nonstationary diffraction problem, i.e., the solution of the equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_x\right)\Gamma(\mathbf{x}, \mathbf{a}, t) = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right)\Gamma(\mathbf{x}, \mathbf{a}, t) = \delta(\mathbf{x} - \mathbf{a}, t)$$

with the same boundary conditions as those imposed above for the stationary diffraction problem. The functions $G(\mathbf{x}, \mathbf{a}, k)$ and $\Gamma(\mathbf{x}, \mathbf{a}, t)$ are related by the formula

$$G(\mathbf{x}, \mathbf{a}, k) = -\int_{-\infty}^{\infty} \Gamma(\mathbf{x}, \mathbf{a}, t) e^{ikt}\, dt.$$

In constructing the Green' s function for the nonstationary diffraction problem, a key role is played by constructing this function for the basic elements of the polyhedron, i.e., for a dihedral angle and for a polyhedral angle. The transition from the Green' s function constructed for these simplest objects to the Green' s function for an arbitrary polyhedron S is carried out in the same way as in the analogous two-dimensional problem (see ([1]), § 2).

The construction of the Green' s function for a dihedral angle is given in ([2]). Thus, if we construct the Green' s function for a polyhedral angle S, we shall be able to pass to the diffraction problem on an arbitrary polyhedron.

2. Introduce a "spherical" coordinate system: $\mathbf{x} = r\vec{\omega}$, where $r > 0$, $|\vec{\omega}| = 1$, $\mathbf{a} = R\vec{\Omega}$, $R > 0$, $|\vec{\Omega}| = 1$. We shall assume that the vertex A of the polyhedral angle S is located at the center of our coordinate system.

Then, for $t < r + R$, i.e., before the front of the wave scattered by the vertex A, the Green' s function can be constructed using the results of ([2]). The problem arises of determining the Green' s function for $t > r + R$, i.e., behind the front of the spherical wave scattered by the vertex A.

The latter problem has no closed-form solution even in the simpler case when the source is at an infinite distance from the vertex of the polyhedral angle, i.e., in the case of incidence of a plane wave. However, in ([3]) the solution of the nonstationary problem of diffraction of a plane wave on a polyhedral angle is reduced to the solution of the Dirichlet problem for the Laplace equation in a bounded domain. The geometric restrictions on the form of the polyhedral angle imposed in that paper (see ([3]), § 4) can be removed by using the results of ([2]).

We shall now show that, knowing the singularities on the front of the wave scattered by the vertex of the angle for the case of incidence of a plane wave arriving in some direction $\vec{\Omega}_0$, we can determine all singularities of the Green' s function $\Gamma(r, \vec{\omega}, R, \vec{\Omega}_0, t)$ on the front of the wave scattered by the vertex of the angle.

Let us recall that, in order to determine the asymptotics of the Green' s function of the stationary diffraction problem, it is necessary to know only the singularities of the Green' s function of the nonstationary problem.

3. It is easy to show that, in a neighborhood of the front of the wave scattered by the vertex of a polyhedral angle (the equation of the front of this wave has the form $t = r + R$), the nonanalytic part of the Green' s function has a ray expansion (on the ray method see [(4)]):

$$\sum_{k=0}^{\infty} \frac{(t - r - R)^{k+k_0}}{\Gamma(k + k_0 + 1)}\, w_k(r, \vec{\omega}, R, \vec{\Omega}), \tag{1}$$

convergent in some neighborhood of the wave front, whose radius tends to zero when approaching the boundary of shadow and light. We shall find a representation of the functions w_k in terms of the coefficient of the leading term of the ray expansion—the function $w_0(r, \vec{\omega}, R, \vec{\Omega})$—and show that this function is determined through the leading term of an analogous ray expansion for the solution of the problem of diffraction of a plane wave.

Since $\Gamma(r, \vec{\omega}, R, \vec{\Omega}, t)$ satisfies the wave equation as a function of $r, \vec{\omega}, t$, it is easy to verify, using the usual arguments of the ray method, i.e. substituting the series (1) into the wave operator and comparing coefficients of equal powers of $t - r - R$, that the function w_k expands in a series in inverse powers of r, from the power r^{-1} to the power r^{-k-1}, and that

$$w_0(r, \vec{\omega}, R, \vec{\Omega}) = r^{-1} w_0(\vec{\omega}, R, \vec{\Omega}).$$

But, by the reciprocity theorem, $\Gamma(r, \vec{\omega}, R, \vec{\Omega}, t)$ also satisfies the wave equation as a function of $R, \vec{\Omega}, t$, so that the w_k also expand in a series in inverse powers of R, and

$$w_0(r, \vec{\omega}, R, \vec{\Omega}) = r^{-1} R^{-1} w_0(\vec{\omega}, \vec{\Omega}).$$

Since $\Gamma(r, \vec{\omega}, R, \vec{\Omega}, t)$ is a homogeneous function of r, R, t of dimension -2, we obtain that the series (1) actually has the form

$$\sum_{k,l=0}^{\infty} \frac{(t-r-R)^{k+l} w_{k,l}(\vec{\omega}, \vec{\Omega})}{\Gamma(k+l+1) r^{k+1} R^{l+1}}. \tag{2}$$

This expression must satisfy the wave equation with respect to the variables $r, \vec{\omega}, t$. Substituting it into the wave operator, we obtain the following relation between $w_{k,l}$ and $w_{k-1,l}$:

$$w_{k,l}(\vec{\omega}, \vec{\Omega}) = -\frac{[k(k-1) + \Delta_\omega]\, w_{k-1,l}(\vec{\omega}, \vec{\Omega})}{2k}. \tag{3}$$

And since the series (2) satisfies the wave equation with respect to the variables $R, \vec{\Omega}, t$, we have

$$w_{k,l}(\vec{\omega}, \vec{\Omega}) = -\frac{[l(l-1) + \Delta_\Omega]\, w_{k,l-1}(\vec{\omega}, \vec{\Omega})}{2l}. \tag{4}$$

Here Δ_ω (Δ_Ω) is the angular part of the Laplace operator in the variables $r, \vec{\omega}$ ($R, \vec{\Omega}$). In the case of spherical coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta\cos\varphi$, $x_3 = r\sin\theta\sin\varphi$,

$$\Delta_\omega = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \operatorname{ctg}\theta\,\frac{\partial}{\partial\theta}.$$

Expressions (3) and (4) make it possible to express all functions $w_{k,l}$ in terms of the initial function $w_{0,0}$. To determine $w_{0,0}$, after multiplying series (2) by R, we pass to the limit as $R \to \infty$, which gives us the solution of the diffraction problem in the case of incidence of a plane wave. In this case, according to (3), for the field inside the front of the wave scattered by the vertex of the polyhedral angle, we obtain a representation in terms of the solution of the Dirichlet problem for the Laplace equation. Thus, by solving this problem numerically, if necessary, we can find the first term of the ray expansion, i.e., the function $w_{0,0}(\vec{\omega}, \vec{\Omega})$.

4. In order to use formulas (3) and (4) to determine $w_{k,l}(\vec{\omega},\vec{\Omega})$, we must apply the operators Δ_ω and Δ_Ω to $w_{0,0}(\vec{\omega},\vec{\Omega})$. But in the case of the numerical determination of $w_{0,0}$, this function is known only for fixed $\vec{\Omega}$ (the direction of the source) and for all $\vec{\omega}$ (the direction of the observation point). Therefore the determination of $w_{k,l}(\vec{\omega},\vec{\Omega})$, generally speaking, causes difficulties. However, in the case of the boundary conditions $u=0$ or $\partial u/\partial n = 0$,

$$\Delta_\Omega w_{0,0}(\vec{\omega},\vec{\Omega}) = \Delta_\omega w_{0,0}(\vec{\omega},\vec{\Omega}). \tag{5}$$

This assertion is a consequence of the following fact. Let $u(r,\vec{\omega},\vec{\Omega},t)$ be the solution of the problem of diffraction of a plane wave traveling in the direction $\vec{\Omega}$ ($r,\vec{\omega}$ are the "spherical" coordinates of the observation point). Then

$$\Delta_\omega u(r,\vec{\omega},\vec{\Omega},t) = \Delta_\Omega u(r,\vec{\omega},\vec{\Omega},t). \tag{6}$$

We shall prove the latter assertion, for simplicity, only for the case when for $t<0$ the function $u(r,\vec{\omega},\vec{\Omega},t)$ has the form of a plane wave (i.e., for $t<0$ the front of the incident wave does not intersect the polyhedral angle S), although it is valid also in the general case.

It is easy to show that $\Delta_\omega u$, as well as $\Delta_\Omega u$, satisfies on S the same boundary conditions as u. Therefore, by the uniqueness theorem, it is enough to prove formula (6) only for $t<0$, when $u(r,\vec{\omega},\vec{\Omega},t)$ has the form of a plane wave:

$$u = f(t - r(\vec{\omega},\vec{\Omega})),$$

where $f(\xi)=0$ for $\xi<0$.

But the function $f(t-r(\vec{\omega},\vec{\Omega}))$ satisfies the wave equation both as a function of $t,r,\vec{\omega}$, and as a function of $t,r,\vec{\Omega}$. Therefore

$$\Delta_\omega u = \left(r^2 \frac{\partial^2}{\partial t^2} - r^2 \frac{\partial^2}{\partial r^2} - 2r\frac{\partial}{\partial r} \right) u = \Delta_\Omega u,$$

which proves (5).

5. Formula (5) enables us, instead of (3) and (4), to give a single representation of $w_{k,l}(\vec{\omega},\vec{\Omega})$ in terms of $w_{0,0}(\vec{\omega},\vec{\Omega})$:

$$w_{k,l}(\vec{\omega},\vec{\Omega}) = \frac{(-1)^{k+l}}{2^{k+l}k!\,l!} \prod_{\sigma=1}^{k} (\sigma(\sigma-1)+\Delta_\omega) \cdot \prod_{s=1}^{l} (s(s-1)+\Delta_\omega) w_{0,0}(\vec{\omega},\vec{\Omega}) \tag{7}$$

The arguments presented carry over entirely to the case of the Green' s function for diffraction on a smooth cone. It is only necessary to require such

the position of the source so that no shadow is formed behind the cone. Let us also note that these arguments apply to boundary conditions of the third kind, $\alpha\,\partial u/\partial n + \beta\,\partial u/\partial t = 0$. Here, however, $\Delta_\omega w_{0,0} \neq \Delta_\Omega w_{0,0}$. This latter problem is meaningful also in the two-dimensional case. In this case the solution of the nonstationary problem of diffraction of a plane wave is known (see (5)), while the Green' s function for the nonstationary problem is unknown.

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Note: Figure translations are in progress. See original paper for figures.