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Abstract

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N. P. KORNEICHUK

THE EXACT VALUE OF BEST APPROXIMATIONS AND WIDTHS OF CERTAIN CLASSES OF FUNCTIONS

(Presented by Academician A. N. Kolmogorov on 14 I 1963)

1. Let $C_{2\pi}$ be the space of continuous functions of period 2π with norm $\|f\| = \max_x |f(x)|$, and let $E_n(f)$ be the best uniform approximation of the function f by trigonometric polynomials of order not exceeding n. If $\mathfrak{M}$ is some set of functions, then we shall put

$$E_n(\mathfrak{M}) = \sup_{f \in \mathfrak{M}} E_n(f).$$

Denote by $W^{(r)}H_\omega$ $(r = 0, 1, 2, ...)$ the class of functions $f \in C_{2\pi}$ for which the derivative of order r has modulus of continuity $\omega(f^{(r)}; t)$ not exceeding a given modulus of continuity $\omega(t)$. For $\omega(t) = Kt^\alpha$ $(0 \le t \le \pi,\ 0 < \alpha \le 1)$ we shall write $W^{(r)}KH^{(\alpha)}$.

In the papers (^{1,2}) we gave the solution of Favard' s problem (^{3}) for the classes $W^{(0)}H_\omega = H_\omega$ and $W^{(1)}H_\omega$, when $\omega(t)$ is a convex function; namely, it was shown that under this condition

$$E_{n-1}(H_\omega) = \frac{1}{2}\,\omega\left(\frac{\pi}{n}\right) \qquad (n = 1, 2, ...); \tag{1}$$

$$E_{n-1}(W^{(1)}H_\omega) = \frac{1}{4}\int_0^{\pi/n} \omega(t)\,dt \qquad (n = 1, 2, ...). \tag{2}$$

To prove the equalities (1) and (2), we first obtained an exact estimate of the best approximations of functions of the classes H_ω and $W^{(1)}H_\omega$ by functions of the classes $KH^{(1)}$ and $W^{(1)}KH^{(1)}$, respectively, which then, for a suitably chosen (extremal) K, were approximated by trigonometric polynomials. In doing so, the known (^{3,4}) equality was used

$$E_{n-1}(W^{(r)}KH^{(1)}) = \frac{4K}{\pi n^{r+1}} \sum_{m=0}^{\infty} \frac{(-1)^{mr}}{(2m+1)^{r+2}} \qquad (n = 1, 2, \dots;\ r = 0, 1, 2, \dots). \tag{3}$$

This method of ours, which led to the exact constant in the estimate of the quantities $E_{n-1}(W^{(r)}H_{\omega})$ $(n = 1, 2, \dots;\ r = 0, 1)$, makes it possible (under the condition of convexity of $\omega(t)$) to obtain the same result in the case $r = 2, 3$. For the widths (^{5,6}) of the classes under consideration this gives an upper estimate which, as it was possible to show with the aid of a theorem of A. N. Kolmogorov, turned out to be exact.

2. Theorem 1. Let $\omega(t)$ be an arbitrary modulus of continuity and let K be any positive number. If $f \in W^{(2)}H_{\omega}$, then

$$\inf_{\varphi \in W^{(2)}KH^{(1)}} \|f - \varphi\| \leq \frac{1}{8} \max_{x \geq 0} \int_0^x t\,|\omega(t) - Kt|\,dt. \tag{4}$$

Theorem 2. If $\omega(t)$ is a convex modulus of continuity, then for all $n = 1, 2, \dots$ there exists a positive number $K_n = K(n)$ such that,

that

$$\sup_{f \in W^{(2)}H_{\omega}} \ \inf_{\varphi \in W^{(2)}K^{(1)}_{nH}} \|f - \varphi\| = \frac{1}{8} \int_0^{\pi/n} t\,|\omega(t) - K_n t|\,dt. \tag{5}$$

The proof of Theorems 1 and 2 is based on the following lemma.

Lemma. Let the function f have a second derivative f'' belonging to the space $C_{2\pi}$. Suppose further that a number $K > 0$ is given such that $f'' \in KH^{(1)}$. Then, in order that for a function $\varphi_0 \in W^{(2)}KH^{(1)}$ the equality

$$\|f - \varphi_0\| = \inf_{\varphi \in W^{(2)}KH^{(1)}} \|f - \varphi\| = \rho$$

hold, it is necessary that there exist at least two points a and b, $a < b$, at which the following conditions are fulfilled:

1) $f(a) - \varphi_0(a) = \varphi_0(b) - f(b) = \pm\rho$;
2) $\varphi_0'(b) - \varphi_0'(a) = K(b - a)\,\mathrm{sign}[f(a) - \varphi_0(a)]$.

With the aid of Theorem 2 and equality (3) one proves

Theorem 3. If $\omega(t)$ is a convex modulus of continuity, then

$$E_{n-1}(W^{(2)}H_\omega) = \frac{1}{8}\int_0^{\pi/n} t\omega(t)\,dt \qquad (n = 1, 2, \dots). \tag{6}$$

Corollary.

$$E_{n-1}(W^{(2)}KH^{(\alpha)}) = \frac{K\pi^{2+\alpha}}{8(2+\alpha)n^{2+\alpha}} \qquad (n = 1, 2, \dots,\ 0 < \alpha < 1).$$

It is not difficult to indicate in the class $W^{(2)}H_\omega$ a function $f_{n2}(x)$ whose best approximation $E_{n-1}(f_{n2})$ is exactly equal to the right-hand side of (6). Let, for $n = 1, 2, \dots;\ r = 0, 1, 2, \dots$, $f_{nr}(x) = f_{nr}(\omega; x)$ be a function of period $2\pi/n$, with mean value over the period equal to zero, for which the derivative of order r is defined by the equalities

$$f_{nr}^{(r)}(x) = \begin{cases} \frac{1}{2}\,\omega(2x), & \left(0 \le x \le \frac{\pi}{2n}\right), \\ \frac{1}{2}\,\omega\left(\frac{2\pi}{n} - 2x\right), & \left(\frac{\pi}{2n} \le x \le \frac{\pi}{n}\right), \end{cases}$$

$$f_{nr}^{(r)}(x) = -f_{nr}^{(r)}(-x).$$

It is easy to verify that for convex $\omega(t)$, $f_{nr} \in W^{(r)}H_\omega$ and $E_{n-1}(f_{nr}) = \|f_{nr}\|$, and in the case $r = 2$, $\|f_{nr}\|$ coincides with the right-hand side of (6).

Results analogous to Theorems 1, 2, and 3 were also obtained by us for the case $r = 3$. With the aid of the functions f_{nr}, relations (1), (2), (6), and also the corresponding equality for the class $W^{(3)}H_\omega$, can be written in the form

$$E_{n-1}(W^{(r)}H_\omega) = \|f_{nr}\| \qquad (n = 1, 2, \dots;\ r = 0, 1, 2, 3), \tag{7}$$

under the assumption that the modulus of continuity $\omega(t)$ is convex. We note that for $n = 1$ equality (7) was established $(^7)$ for all $r = 0, 1, 2, \dots$.

In the case when $\omega(t)$ is not a convex function, our estimates of the quantities $E_{n-1}(W^{(r)}H_\omega)$ are less precise. We give some results for $r = 2$. Put

$$\eta(x, K) = \int_0^x t\,|\omega(t) - Kt|\,dt$$

and let X_ω be the set of points x_0 of the half-line $[0, +\infty)$ for which, for some $K_0 = K(x_0)$, the equality

$$\max_{x \ge 0} \eta(x, K_0) = \eta(x_0, K_0) \tag{8}$$

holds.

From Theorem 1 and equality (3), for arbitrary $\omega(t)$ we obtain the estimate

$$E_{n-1}\left(W^{(2)}H_\omega\right) \leqslant \frac{1}{8}\left\{\int_0^{\pi/n} t\omega(t)\,dt + \int_{\pi/n}^{x_0} t\left[\omega(t) - K_0 t\right]dt\right\}, \tag{9}$$

where x_0 is any point of the set X_0, and the number K_0 is related to x_0 by equality (8).

If for the modulus of continuity $\omega(t)$ there exist natural numbers $n_1 < n_2 < n_3 < \cdots$ such that $\pi/n_k \in X_0$, then

$$E_{n_k-1}\left(W^{(2)}H_\omega\right) \leqslant \frac{1}{8}\int_0^{\pi/n_k} t\omega(t)\,dt \qquad (k = 1, 2, \ldots).$$

If $\pi/n \notin X_0$, then naturally in estimate (9) one chooses $x_0 \in X_0$ so that the second integral assumes the least value.

3. Let $d_n(\mathfrak{M})$ be the n-th width of the set $\mathfrak{M}$ in the space $C_{2\pi}$, i.e., the lower bound of the deviations of the set $\mathfrak{M}$ from all possible manifolds of dimension $\leqslant n$. The notion of width for functional classes was introduced by A. N. Kolmogorov[5]. V. M. Tikhomirov, approximating functions of the class H_ω by piecewise constant functions, showed (see Theorem 3 in [6] and the correction in the abstract of that paper (RZhMat, 1962, 4B69)) that, for a convex modulus of continuity $\omega(t)$,

$$d_{2n-1}(H_\omega) = \frac{1}{2}\omega\left(\frac{\pi}{n}\right) = \|f_{n0}\| \qquad (n = 1, 2, \ldots). \tag{10}$$

From equality (7) it follows that

$$d_{2n-1}\left(W^{(r)}H_\omega\right) \leqslant \|f_{nr}\| \qquad (n = 1, 2, \ldots;\ r = 1, 2, 3).$$

To obtain a lower estimate, take on $[0, 2\pi)$ a system of $2n$ points $x_k = k\pi/n$ $(k = 0, 1, 2, \ldots, 2n-1)$. It can be shown that, for fixed $n = 1, 2, \ldots$, whatever the system of indices $i_0, i_1, \ldots, i_{2n-1}$, $i_k = \pm 1$, in each of the classes $W^{(r)}H_\omega$ $(r = 1, 2, 3)$, where $\omega(t)$ is a convex modulus of continuity, there exists a function $f_r(x)$ such that $f_r(x_k) \geqslant \|f_{nr}\|$ if $i_k = +1$, and $f_r(x_k) \leqslant -\|f_{nr}\|$ if $i_k = -1$.

By virtue of A. N. Kolmogorov' s theorem (see [6], Theorem 2), from this we conclude that

$$d_{2n-1}\left(W^{(r)}H_\omega\right) \geqslant \|f_{nr}\| \qquad (n = 1, 2, \ldots;\ r = 1, 2, 3).$$

Thus, the following holds.

Theorem 4. *If $\omega(t)$ is a convex modulus of continuity, then*

$$d_{2n-1}\left(W^{(r)}H_\omega\right) = \|f_{nr}\| \qquad (n = 1, 2, \ldots;\ r = 1, 2, 3). \tag{11}$$

From equalities (7), (10), and (11) it follows that, for each $n = 1, 2, \ldots$, the set of trigonometric polynomials of degree not exceeding $n-1$ is an extremal subspace of dimension $2n-1$ for the classes $W^{(r)}H_\omega$ $(r = 0, 1, 2, 3)$ when $\omega(t)$ is convex.

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Note: Figure translations are in progress. See original paper for figures.

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