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Abstract

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THE SET OF LIMIT VALUES OF PSEUDOANALYTIC FUNCTIONS IN THE UNIT DISK*

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Let $w = f(z)$ be a pseudoanalytic function defined in the disk $D: \ |z| < 1$ (a many-sheeted quasiconformal mapping of the disk D). Let A be some arc of the circle $\Gamma: \ |z| = 1$, and let E be a closed set on A. To each point $e^{i\theta} \in A \setminus E = \mathcal{C}E$ we assign a certain (arbitrary) Jordan curve Λ_θ, lying in D and ending at $e^{i\theta}$, and denote by $C_{\Lambda_\theta}(f, e^{i\theta})$ the set of limit values of the function $f(z)$ at the point $e^{i\theta}$ along the curve Λ_θ. As usual, $C_D(f, e^{i\theta})$ is the set of all limit values of the function $f(z)$ at the point $e^{i\theta}$; $R_D(f, e^{i\theta})$ is the aggregate of values assumed by the function $f(z)$ infinitely often in every neighborhood of the point $e^{i\theta}$. At an arbitrary point $z_0 = e^{i\theta_0} \in E$ we define the set $C_{\Gamma\setminus E}(f, z_0)$ as follows:

$$C_{\Gamma\setminus E}(f, z_0) = \bigcap_{\eta>0} M_\eta,$$

where M_η is the closure of the union

$$\bigcup_{\theta} C_{\Lambda_\theta}(f, e^{i\theta})$$

over all points $e^{i\theta} \in \mathcal{C}E \cap \{|e^{i\theta} - z_0| < \eta\}$.

Consider the set $\Omega = C_D(f, z_0) \setminus C_{\Gamma\setminus E}(f, z_0)$.

The aim of the present note is to prove the following theorem.

Theorem. a) The set Ω is open;

b) if the set Ω is nonempty, and Ω_n is any connected component of it, then $\Omega_n \setminus R_D(f, z_0)$ consists of at most two points;

c) if for some n_0 the set $\Omega_{n_0} \setminus R_D(f, z_0) = \{w_0, w_1\}$, $w_0 \neq w_1$, then the set $R_D(f, z_0)$ coincides with the extended w-plane from which the points w_0, w_1 have been removed;

d) every value $\alpha \in \Omega \setminus R_D(f, z_0)$ is an asymptotic value of the function $f(z)$ either at the point z_0, or at each point of some sequence of points $\{z_n\}$, $z_n \in \Gamma$, converging to z_0.

Assertion c) of the theorem refines the theorem of Størvick (^{1}) and is a generalization, to the case of pseudoanalytic functions, of a result of Wolff (^{2}), established for meromorphic functions when the curves Λ_θ are radii of the disk D. Assertion d) of the theorem was proved by Størvick (^{1}) in the case when the Λ_θ are radii of the disk D. The method of our proof differs from the methods of (^{1,2}) and is close to the methods of Noshiro (^{3}), who established an analogous theorem for meromorphic functions under a weaker restriction on the set E: Noshiro assumed $\operatorname{mes} E = 0$. However, the proofs in (^{3}) do not go through in the case of pseudoanalytic functions. Therefore a stronger restriction had to be imposed on the set E ($\operatorname{cap} E = 0$), and in addition to the methods of (^{3}) several modernized methods from other works of Noshiro (^{4,5}) had to be used.

We establish the validity of assertion a) of the theorem. Let $w_0 \in \Omega$; choose $\eta > 0$ so that the point w_0 lies outside M_η at a distance ρ. In the neighborhood $U(z_0, \eta):\ |z - z_0| < \eta$, consider two points $e^{i\theta_1}, e^{i\theta_2} \in \mathcal{C}E$ ($\theta_1 < \theta_0 < \theta_2$), and join by a straight-line segment s the final parts of the curves $\Lambda_{\theta_1}, \Lambda_{\theta_2}$ that have fallen into $U(z_0, \eta)$. Denote the resulting curve by l, and the domain bounded by the curve l and the arc $e^{i\theta}$, $\theta_1 \leqslant \theta \leqslant \theta_2$), through D_1. We may assume that the image l under $w = f(z)$ is at a positive distance ρ ($\rho \leqslant \rho'$) from M_η. Since $w_0 \in C_D(f, z_0)$, there exists a sequence of points $z_\mu \in D_1$, $z_\mu \to z_0$, $f(z_\mu) \to w_0$, $\mu \to \infty$. We shall regard the sequence $\{z_\mu\}$ as fixed throughout the proof. The preimage of the disk $|w - w_0| < \rho$ in the domain D_1 consists of at most a countable number of connected components; denote by Δ_μ the component of the preimage containing z_μ.

First consider the case when the number of distinct components Δ_μ is infinite. We may assume that $\Delta_\mu \neq \Delta_\nu$, $\mu \neq \nu$. Then the sequence $\{\Delta_\mu\}$ converges to the point z. Otherwise one could choose points $z_1' = e^{i\theta_1'}$, $z_2' = e^{i\theta_2'}$, $z_1', z_2' \in \mathcal{C}E$ ($\theta_1 < \theta_1' < \theta_0 < \theta_2' < \theta_2$) and construct for them, by the method indicated above, a curve l' such that the intersection $l' \cap \Delta_{\mu_n}$, $n = 1, 2, \ldots$, would be nonempty. Let ζ_n be a point of intersection of the curve l' and the boundary of Δ_{μ_n}, and let ζ_0 be a limit point of the sequence $\{\zeta_n\}$; then $\zeta_0 \in \mathcal{C}E$ or $\zeta_0 \in D$. The case $\zeta_0 \in \mathcal{C}E$ is impossible, because then the set M_η would intersect l^*: $|w - w_0| = \rho$ ($f(\zeta_n) \in \{|w - w_0| = \rho\}$). The case $\zeta_0 \in D$ is also impossible, since then an arbitrarily small neighborhood of the point ζ_0 would intersect infinitely many level lines $|f(z) - w_0| = \rho$.

If Δ_μ is compact in D, then $w = f(z)$ assumes in Δ_μ every value from the disk $|w - w_0| \leqslant \rho$. Let Δ_μ be noncompact in D. We shall show that the domain Δ_μ is locally connected on the boundary; otherwise there exists a sequence $\{\Gamma_n\}$ of boundary contours of the domain Δ_μ, converging to some arc C on Γ in

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the neighborhood $U(z_0, \eta)$, and, consequently, the sets $C_{\Delta\theta}(f, e^{i\theta})$ at the points $e^{i\theta} \in \mathcal{C}E \cap C$ would intersect the curve l^*, i.e. $M_\eta \cap l^*$ would be nonempty, which contradicts the choice of M_η. We shall show that the part of the boundary of the domain Δ_μ lying on Γ—the set E_μ—has capacity zero. For this, write $E_\mu = (E_\mu \cap E) \cup (E_\mu \cap \mathcal{C}E)$. By the preceding, through every point $e^{i\theta} \in E_\mu \cap \mathcal{C}E$ one can draw a continuous curve L_θ, lying entirely inside Δ_μ. Therefore the sets $C_{L_\theta}(f, e^{i\theta})$ and $C_{\Delta\theta}(f, e^{i\theta})$ do not intersect. According to Beurling' s theorem (6), there are at most countably many such points $e^{i\theta}$. Consequently, E_μ has capacity zero.

Now we have the right to apply the following lemma (4).

Lemma (4). *Let $w = f(z)$ be a pseudoanalytic function in a bounded domain D, and let E be a closed set of capacity zero on the boundary Γ. If $\overline{\lim}_{z\to\zeta}|f(z)| \leqslant M$ at every point $\zeta \in \Gamma \setminus E$, and $f(z)$ is bounded in a neighborhood of every point of the set E, then $|f(z)| \leqslant M$ at all points of the domain D.*

According to this lemma, in our case the set of values $\mathfrak{D}_\mu = f(\Delta_\mu)$ is everywhere dense in $|w - w_0| < \rho$ and, consequently, its closure $\overline{\mathfrak{D}}_\mu$ coincides with $|w - w_0| \leqslant \rho$. Since $\{\Delta_\mu\}$ converges to z_0, $C_D(f, z_0)$ contains the closed disk $|w - w_0| \leqslant \rho$.

Now consider two monotonically decreasing sequences of positive numbers $\{\eta_n\}$, $\{\rho_n\}$ and the corresponding sequence of curves $\{l_n\}$; for fixed n the number ρ_n and the curve l_n from the neighborhood $|z - z_0| < \eta_n$ are chosen by the method indicated above. Denote by $\Delta_\mu^{(n)}$ the connected component containing z_μ of the preimage of the disk $|w - w_0| < \rho_n$. Suppose that there is at least one index n for which the sequence $\{\Delta_\mu^{(n)}\}$ $(\mu \geqslant N(n))$ consists of infinitely many members. Then, by the preceding, $C_D(f, z_0)$ contains the closed disk $|w - w_0| \leqslant \rho_n$.

It remains to consider the case when, for every n, $\{\Delta_\mu^{(n)}\}$ consists of a finite number of distinct domains. Denote by $\Delta^{(1)}$ some

a domain $\Delta_\mu^{(1)}$ containing an infinite subsequence $\{z_\mu^{(1)}\}$ of the sequence $\{z_\mu\}$; by $\Delta^{(2)}$, some domain $\Delta_\mu^{(2)}$ containing an infinite subsequence $\{z_\mu^{(2)}\}$ of the sequence $\{z_\mu^{(1)}\}$, etc. We obtain a new sequence of domains $\{\Delta^{(n)}\}$, $\Delta^{(1)} \supset \Delta^{(2)} \supset \cdots \supset \Delta^{(n)} \supset \cdots$; all $\Delta^{(n)}$ have on their boundary a common point z_0. Since the set of values of the function $w = f(z)$ in $\Delta^{(n)}$ lies inside $|w - w_0| < \rho_n$, and the diameters of the domains $\Delta^{(n)}$ tend to zero as $n \to \infty$, in D_1 there exists a curve Λ, ending at z_0, along which $w = f(z)$ tends to w_0.*

Denote by Δ the connected component of the preimage of the disk $|w - w_0| < \rho$ that contains the last part of the curve Λ. Just as above, one can show that the boundary of Δ consists of a closed set E_0 of capacity zero on Γ and at most a countable number of curves in D. We are now in the conditions of applicability of the methods used by Noshiro in [7] to prove part (ii) of Theorem 3. By means of these methods it is established that $C_D(f, z_0) \setminus R_\Delta(f, z_0)$ has capacity zero.

To prove assertion c) of the theorem, in the preceding arguments one must replace the point w_0 by the point α.

We shall prove assertion b) of the theorem. We shall give two proofs of this part: one for meromorphic functions, the other for pseudoanalytic functions.

Let the component Ω_{n_0} contain three distinct values $w_0, w_1, w_2 \in \mathscr{C}R_D(f, z_0)$. Through the point w_2 draw a closed analytic curve $\mathscr{L}$ so that its interior G consists entirely of points of the domain Ω_{n_0}, and $w_0, w_1 \in G$. The domain G plays the same role as the disk $|w - w_0| < \rho$ in the preceding proof. As above, we construct the domain D_1 and denote by Δ the component of the preimage of the domain G containing an asymptotic curve Λ, which ends at some point $z_0' = e^{i\theta_0}$, $\theta_1 < \theta_0' < \theta_2$ (possibly $z_0' = z_0$), and along which $f(z) \to w_0$ (the existence of the curve Λ is guaranteed by part c) of the theorem). One may assume that the image of the curve Λ under $w = f(z)$ lies entirely inside G. The component Δ is simply connected. Indeed, the boundary of Δ cannot contain any closed curve in D_1, since otherwise the image of such a curve would lie on the curve $\mathscr{L}$, passing through the exceptional value w_2 of the function $w = f(z)$. The boundary of the domain Δ consists of a finite number of segments q_i on s, at most a countable number of analytic arcs $\{\Gamma_n\}$, $\Gamma_n \in D_1$, and a closed set E_0 on Γ. As above, one can show that E_0 has capacity zero.

Up to this point all the arguments have been valid both for meromorphic functions and for pseudoanalytic functions. Now suppose that $w = f(z)$ is a meromorphic function in D.

By means of the function $z = z(\zeta)$, map the domain Δ conformally onto the unit disk $|\zeta| < 1$. The image of the curve Λ under $z = z(\zeta)$ will be some curve ending at the point ζ_0, $|\zeta_0| = 1$. On $|\zeta| = 1$ consider a sufficiently small arc $A_\zeta \ni \zeta_0$ having no common points with the images, under $z = z(\zeta)$, of the segments q_i. On the arc A_ζ consider the set E_ζ, at each point $\zeta = e^{i\varphi}$ of which both functions $z = z(\zeta)$ and $w = w(\zeta) \equiv f(z(\zeta))$ have definite angular boundary values $z(e^{i\varphi})$, $w(e^{i\varphi})$, and $w(e^{i\varphi}) \in G$. Suppose that there exists a point $e^{i\varphi} \in E_\zeta$ at which $z(e^{i\varphi}) = e^{i\theta} \notin E$. According to Beurling' s theorem, such points $e^{i\varphi}$ are countable– set. Consequently, the set E_ζ has capacity zero, and at all points $e^{i\varphi} \in A_\zeta \setminus E_\zeta$ the angular boundary values $w(e^{i\varphi})$ of the function $w(\zeta)$ lie on $\mathscr{L}$. If by $W = \Phi(w)$ we denote a conformal mapping of the domain G onto the disk $|W| < 1$, then the function

$$W = W(\zeta) = \Phi(f(z(\zeta)))$$

will be a function of class (U) in Seidel' s sense. Indeed, the function $W(\zeta)$ is regular and bounded, $|W(\zeta)| < 1$, $|\zeta| < 1$, and $|W(e^{i\varphi})| = 1$ for $e^{i\varphi} \in A_\zeta \setminus E_\zeta$, where $W(e^{i\varphi})$ is the radial boundary value of the function $W(\zeta)$ at the point $e^{i\varphi}$. Further, the function $W = W(\zeta)$ has at the point $\zeta_0 = e^{i\varphi_0}$ the radial boundary

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* Relying on the results from (4) and on the fact that, by definition, the set $\mathscr{C}R_D(f, z_0)$ contains three distinct points w_0, w_1, w_2, one can show that the point $z_0' \in E$.

value $W(e^{i\varphi_0}) = \Phi(w_0)$, lying in $|W| < 1$. According to Løvatør' s theorem (^{8}), the function $W(\zeta)$ assumes every value from $|W| < 1$ infinitely often in any neighborhood of the point ζ_0, with the exception, perhaps, of one. On the other hand, the function $W(\zeta)$ has two exceptional values (corresponding to the values w_0, w_1). We have arrived at a contradiction. The proof of part b) cannot be transferred to the case when $w = f(z)$ is a pseudoanalytic function. In this case the arguments of (^{4}) apply, based on the properties of the Evans-Zelberg function associated with a set E_0 of capacity zero (the set E_0 is the part of the boundary of the component Δ lying on Γ), and the proof ends in the same way as in (^{4}).

By analogous methods one establishes the validity of assertion c) of the theorem.

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