MATHEMATICS

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1963

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Abstract

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MATHEMATICS

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ELLIPTIC BOUNDARY VALUE PROBLEMS DEPENDING ON A PARAMETER

(Presented by Academician I. G. Petrovsky, 12 X 1962)

In this note we consider a boundary value problem of general form for a system, elliptic in the sense of I. G. Petrovsky, in a bounded domain G of the n-dimensional space R^n. The coefficients of the system and of the boundary operators are assumed to depend, in a certain way, on a parameter $q \in Q$, where Q is an angle in the complex plane with vertex at the origin. Algebraic conditions are indicated (imposed on the system and on the boundary operators) which are sufficient for the unique solvability of the problem for large $|q|$.

Let us write the problem in the following form:

$$Au \equiv \sum_{\alpha+|\beta|\leq s} a_{\alpha\beta}(x) q^\alpha D^\beta u(x) = f(x) \quad \text{in } G, \tag{1}$$

$$B_\nu u \equiv \sum_{\alpha+|\beta|\leq m_\nu} b_{\nu\alpha\beta}(y) q^\alpha D^\beta u(x)\big|_{x=y} = g_\nu(y) \quad (y \in \Gamma,\ \nu = 1, \dots, r). \tag{2}$$

Here $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$; $D^\beta = D_1^{\beta_1} \cdots D_n^{\beta_n}$; $D_j = -i\,\partial/\partial x_j$; α, β_j, and m_ν are nonnegative integers; $|\beta| = \sum \beta_j$. By $u(x)$ and $f(x)$ are denoted column vectors of height N; $a_{\alpha\beta}(x)$ are square matrices of order N; $b_{\nu\alpha\beta}(x)$ are rows of length N. The number Ns will be even; $r = Ns/2$. The reader may, if desired, keep in mind the case of a single equation ($N = 1,\ s = 2m$). The coefficients of the operators A and B_ν, and the boundary Γ of the domain G, are assumed to be infinitely smooth (or sufficiently smooth); the boundary Γ admits local straightening by means of coordinate transformations. The complex parameter q varies in the angle $Q = Q_\theta :\ |\arg q| \leq \theta$; in particular, the case of the ray $\theta = 0$ is possible.

We shall call the system (1) **semi-bounded** in Q if

$$A_\theta(x, q, \xi) \equiv \sum_{\alpha+|\beta|=s} a_{\alpha\beta}(x) q^\alpha \xi^\beta \neq 0 \quad \text{for } x \in \overline{G},\ q \in Q,\ |q| + |\xi| \neq 0. \tag{3}$$

Here $\overline{G}$ is the closure of the domain G; ξ is a real vector $(\xi_1, \dots, \xi_n)$; $\xi^\beta = \xi_1^{\beta_1} \cdots \xi_n^{\beta_n}$; $|\xi|^2 = \sum \xi_j^2$. A semi-bounded system is elliptic: this is seen from (3) when $q = 0$.

Let us note especially the case when q enters into (1) only in powers that are multiples of some number $k > 0$, and $\theta = \pi/2k$, so that $p = q^k$ varies in the right half-plane $\operatorname{Re} p \geq 0$. In this case k will necessarily be an even integer, $k = 2b$, and the semi-boundedness of the system (1) is equivalent to the fact that the system obtained from (1) by replacing p by $\partial/\partial t$ is $2b$-parabolic in the sense of I. G. Petrovsky ([1]).

In particular, the system (1) may have the form

$$A(x, D)u(x) + pu(x) = f(x) \quad (p = q^{2b},\ s = 2b). \tag{4}$$

where $A(x, D)$ does not depend on q. We also note that if $Au = f$ is a strongly elliptic (2) system with real coefficients, then (4) is a semi-bounded system in $Q_{\pi/4b}$ (possibly after changing the sign before p).

Now let us give the definition of semi-boundedness of problem (1)–(2). Let y be any point on Γ. Put

$$B_{\nu 0}(y, q, \xi) = \sum_{\alpha + |\beta| = m_\nu} b_{\nu\alpha\beta}(y) q^\alpha \xi^\beta.$$

Translate the origin of coordinates to y and rotate the system so that the x_n-axis takes the direction of the inner normal to Γ at y. In order not to complicate the notation, suppose that problem (1)–(2) has already been written in this new coordinate system. We agree to write $\xi = (\xi', \xi_n)$. Consider on the ray $x_n > 0$ the problem

$$A_0(y, q, \xi', D_n)v(x_n) = 0, \tag{5}$$

$$B_{\nu 0}(y, q, \xi', D_n)v(0) = h_\nu \quad (\nu = 1, \dots, r). \tag{6}$$

We shall call problem (1)–(2) **semi-bounded** in Q if: 1) the system (1) is semi-bounded in Q, and 2) problem (5)–(6) is uniquely solvable in the class $\mathfrak{M}$ of solutions of system (5) which tend to 0 as $x_n \to +\infty$, for arbitrary h_ν, $q \in Q$, and ξ', $|\xi'| + |q| \neq 0$ (and for any point $y \in \Gamma$).

For $n = 1$, to 1) and 2) one adds the assumption that exactly one half of the roots λ of the equation

$$\det A_0(y, q, \xi', \lambda) = 0 \tag{7}$$

lie in the upper half-plane; because of this, $\mathfrak{M}$ has dimension r. For $n > 1$ this follows from (3) (cf. (3), p. 123).

Condition 2) can be written in algebraic form analogously to the condition of Ya. B. Lopatinskii (3). Condition 2) for $q = 0$ becomes the condition of Ya. B. Lopatinskii (3).

In the case when q enters (1) and (2) only in powers divisible by $2b$, and $\theta = \pi/4b$, semi-bounded problems are closely connected ($q^{2b} \leftrightarrow \partial/\partial t$) with mixed problems for systems parabolic in the sense of I. G. Petrovskii, in cylindrical domains (see (4–6)). This connection and applications of the results presented below are considered in a separate note (15).

We shall consider problem (1)–(2) in the spaces $W_2^{(k)}$ (7, 8). For a nonnegative integer k, denote by $H_k(G)$ the direct product of N spaces $W_2^{(k)}(G)$. The square of the norm $\| \ \|_k$ of a vector in $H_k(G)$ is equal to the sum of the squares of the norms of its components. By $H_{k+1/2}(\Gamma)$ we shall mean the space $W_2^{(k+1/2)}(\Gamma)$; its norm will be denoted by $\| \ \|_{k+1/2}$.

Let l be an integer $\geq \max(s, m_\nu + 1)$. Denote by $H_l(G, \Gamma)$ the direct product of the spaces $H_{l-s}(G)$ and $H_{l-m_\nu-1/2}(\Gamma)$ ($\nu = 1, \dots, r$). For every fixed q, the operator $\mathfrak{A} = (A, B_1, \dots, B_r)$ acts continuously from $H_l(G)$ to $H_l(G, \Gamma)$. It is known (9) that the ellipticity of the system A and the Lopatinskii condition are necessary and sufficient for $\mathfrak{A}$ to be a Φ-operator and for an estimate of the form

$$\|u\|_l \leq C(q) \left\{ \|f\|_{l-s} + \sum_{\nu=1}^{r} \|g_\nu\|_{l-m_\nu-1/2} + \|u\|_0 \right\}. \tag{8}$$

Theorem. *Suppose that problem (1)–(2) is semi-bounded in Q. Then there exists a $q_0 \geq 0$ such that, for every fixed q, $|q| \geq q_0$, $q \in Q$, the operator $\mathfrak{A}$ maps $H_l(G)$ one-to-one onto $H_l(G, \Gamma)$. In other words, for such q problem (1)–(2), for arbitrary*

$f \in H_{l-s}(G)$ and $g_\nu \in H_{l-m_\nu-1/2}(\Gamma)$ has one and only one solution $u \in H_l(G)$. Moreover, the two-sided estimate holds

$$\|u\|_l + |q|^l \|u\|_0 \leq C_1 \{\|f\|_{l-s} + |q|^{l-s}\|f\|_0 + \sum_{\nu=1}^{r} [\|g_\nu\|_{l-m_\nu-1/2} + |q|^{l-m_\nu-1/2}\|g_\nu\|_0]\} \leq C_2\{\|u\|_l + |q|^l\|u\|_0\}, \tag{9}$$

where the constants C_1 and C_2 depend neither on the functions u, f, g_ν, nor on q.

Corollary. If (1)–(2) is a semi-bounded problem in the ray $Q : \arg q = 0$, then this problem is uniquely solvable for all complex q, except for a discrete set of points q.

Remark 1. It can be shown that the Dirichlet problem for a system of the form (4), where $Au = f$, a strongly elliptic system with real coefficients, is semi-bounded in $Q_{\pi/4b}$ (possibly after changing the sign before ρ). The results formulated above may be regarded as a generalization of the well-known theorem ([2,10]) that this problem is uniquely solvable for all complex ρ, except for a discrete set of points ρ.

Remark 2. If for the problem (1)–(2) an estimate of the form (9) holds for $q \in Q$, $|q| \geq q_0$, then this problem is semi-bounded in Q.

We outline the proof of the theorem. Let $y \in G$. Consider in the whole space R^n the "homogeneous" system with coefficients frozen at the point y,

$$A_0(y, q, D)u(x) = f(x). \tag{10}$$

By virtue of (3), for $q \neq 0$ and $f \in H_{l-s}(R^n)$ this system is uniquely solvable in $H_l(R^n)$, and the solution admits the representation

$$u(x) = R(y)f(x) = F^{-1}A_0^{-1}(y, q, \xi)(Ff)(\xi), \tag{11}$$

where F is the Fourier transform $x \to \xi$.

Now let $y \in \Gamma$. Again assuming that y is the origin of coordinates and that the axis x_n is directed along the interior normal to Γ at y, consider in the half-space $R^n_+ : x_n > 0$ the "homogeneous" problem with coefficients frozen at the point y, (10),

$$B_{\nu 0}(y, q, D)u(x)\big|_{x_n=0} = g_\nu(x') \quad (\nu = 1, \dots, r). \tag{12}$$

Here $x = (x', x_n)$, $x' = (x_1, \dots, x_{n-1}) \in R^{n-1}$. From semi-boundedness it follows that, for $q \neq 0$, $f \in H_{l-s}(R^n_+)$, $g_\nu \in H_{l-m_\nu-1/2}(R^{n-1})$, the problem (10), (12) is uniquely solvable in $H_l(R^n_+)$. The solution admits the representation

$$u(x) = R(y)(f, \dot{g}) = R_0(y)f + \sum_{k=1}^{r} R_k(y)\,[g_k - B_{k0}(y)R_0(y)f]. \tag{13}$$

Here $R_0(y)f$ is a particular solution of the system (10), defined by a formula of the form (11), in which f must be replaced by Lf; L is an extension operator from R^n_+ to R^n, bounded in the norms $\|\ \|_k$, $k \leq l - s$. Further, $R_k(y)g_k$ is the solution of the problem (10), (12) with $f = 0$ and $g_\nu = 0$ for $\nu \neq k$. One can show that

$$R_k(y)g_k = F'^{-1}\left\{\int_C e^{i\lambda x_n} A_0^{-1}(y, q, \xi', \lambda)P_k(y, q, \xi', \lambda)\, d\lambda \cdot (F' g_k)(\xi')\right\}, \tag{14}$$

where F' is the Fourier transform $x' \to \xi'$; $P_k(y, q, \xi', \lambda)$ is a column of height N consisting of functions, infinitely smooth and positively homogeneous in (q, ξ', λ) of degree $s - m_k - 1$, which are polynomials in λ (cf. (9)).

These formulas make it possible to obtain estimates of the required form for $\|u\|_l + |q|^l \|u\|_0$ in R^n and in R^n_+ for constant coefficients and "homogeneous"

operators for all $q \neq 0$ from Q. Hence, by means of the well-known localization method, for sufficiently large $|q|$ the first of inequalities (9) is derived for problem (1)–(2) in G. The second of inequalities (9) is verified directly. In these considerations the inequalities

$$|q|^k \|u\|_{l-k} \leq C_{lk} \{\|u\|_l + |q|^l \|u\|_0\}, \tag{15}$$

are used, where $k = 1, \ldots, l-1$, and where the constants C_{lk} do not depend on $u(x)$ and q.

From (9) follows the uniqueness of the solution of problem (1)–(2). To prove existence one constructs an operator R, depending on q, "almost inverse" to the operator $\mathfrak{A}$. More precisely: introduce in the space $H_l(G, \Gamma)$ the norm

$$\|(f, g)\| = \|f\|_{l-s} + |q|^{l-s} \|f\|_0 + \sum_\nu \{\|g_\nu\|_{l-m_\nu-1/2} + |q|^{l-m_\nu-1/2} \|g_\nu\|_0\}.$$

Then, for each $q \neq 0$ from Q, the operator R acts continuously from $H_l(G, \Gamma)$ into $H_l(G)$; for sufficiently large $|q|$ the norm of the operator $\mathfrak{A}R - I$ in $H_l(G, \Gamma)$ (where I is the identity operator) is less than 1. Consequently, for such $|q|$ there exists in $H_l(G, \Gamma)$ a bounded operator T such that $\mathfrak{A}R \cdot T = I$. The solution of problem (1)–(2) is determined by the formula $u = RT(f, g)$. The operator R is constructed as follows:

$$R(f, g) = \sum_k \psi_k(x) R(x_k)(\varphi_k f, \varphi_k g). \tag{16}$$

Here $\varphi_k(x)$ and $\psi_k(x)$ are infinitely differentiable functions with sufficiently small supports; $\sum \varphi_k(x) \equiv 1$ in $\overline{G}$, and $\varphi_k \psi_k = \varphi_k$; x_k is a point belonging to the support of the function φ_k. By $R(x_k)$ is denoted the inverse operator constructed with frozen coefficients. If the boundary Γ does not intersect the support of $\varphi_k(x)$, then $R(x_k)$ is determined by a formula of the form (11). In the opposite case, x_k is chosen on Γ, and $R(x_k)$ is determined by a formula of the form (13) in local coordinates.

Remark 2 is proved by means of examples analogous to those given in ([11]).

Remark 3. After this work had been completed, it became known to us that Sh. Agmon and L. Nirenberg recently obtained ([12,13]) similar results under the following assumptions: $N = 1$ (one equation); the orders of the boundary operators $m_\nu < 2m = s$ $(\nu = 1, \dots, m = r)$; $l = 2m$; $g_\nu(y) = 0$ $(\nu = 1, \dots, m)$. The methods of Agmon and Nirenberg differ from ours; moreover, the method of proof of existence is apparently substantially connected with the restrictions just indicated.

Remark 4. *For semi-bounded systems on a compact surface without boundary* (in this case there are no boundary conditions; cf. ([14])) *there are valid analogues of our theorem, its corollary, and Remark 2.*

Received
8 X 1962

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Note: Figure translations are in progress. See original paper for figures.