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Abstract

Full Text

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ON A PROBLEM OF HUA LOO-KENG

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Hua Loo-keng in ([1]) (Ch. XII, p. 170) poses the following problem: It is well known that the set of integer points lying on the line $ax+b$, $(a,b)=1$, $x=1,2,...$, contains infinitely many primes. Does the set of integer points on the plane

$$(ax+by+c,\ a'x+b'y+c')$$

contain infinitely many pairs of primes (p_1,p_2)?

In the present note it will be shown that this problem is answered affirmatively and is a trivial consequence of the Hardy-Littlewood circle method in the case when

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \neq 0.$$

Introduce the sum

$$S\left(N, \begin{matrix} a & b & c \\ a' & b' & c' \end{matrix}\right) = \sum_{x=-\infty}^{\infty} \sum_{y=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \exp\left[-\frac{(|b|+|b'|)y+n+m}{N}\right] \Lambda(n)\Lambda(m), \tag{1}$$

$$ax+by+c=n, \qquad a'x+b'y+c'=m.$$

If it is shown that as $N\to\infty$ this sum also tends to ∞, then thereby an affirmative answer to Hua Loo-keng' s question will be obtained.

Transform our sum as follows:

$$S\left(N, \begin{matrix} a & b & c \\ a' & b' & c' \end{matrix}\right) = \sum_{x=-\infty}^{\infty} \sum_{y=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \exp\left[-\frac{(|b|+|b'|)y+n+m}{N}\right] \Lambda(n)\Lambda(m),$$

$$x=\frac{n-c-by}{a}, \qquad n\equiv c+by \pmod{a},$$

$$x=\frac{m-c'-b'y}{a'}, \qquad m\equiv c'+b'y \pmod{a'}.$$

But since x ranges over $(-\infty,\infty)$, our sum may be written in the form

$$S\left(N, \begin{matrix} a & b & c \\ a' & b' & c' \end{matrix}\right) = \sum_{y=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \exp\left[-\frac{(|b|+|b'|)y+n+m}{N}\right] \Lambda(n)\Lambda(m),$$

$$a'(n-c-by)=a(m-c'-b'y),$$
$$n\equiv c+by \pmod a,\qquad m\equiv c'+b'y \pmod{a'}.$$

Introduce into the sum discontinuous factors in the form of the known trigonometric sums for our congruences:

$$S\left(N,\begin{matrix}a & b & c\\ a' & b' & c'\end{matrix}\right)=\frac{1}{aa'}\sum_{\nu=0}^{a-1}\sum_{\nu'=0}^{a'-1}\exp\left[-2\pi i\left(\frac{c\nu}{a}+\frac{c'\nu'}{a'}\right)\right]\times$$
$$\times\sum\sum\sum\exp\left\{-\frac{(|b|+|b'|)y+n+m}{N}+2\pi i\left[\frac{\nu}{a}(n-by)+\frac{\nu'}{a'}(m-b'y)\right]\right\}\Lambda(n)\Lambda(m),$$
$$a'(n-c-by)=a(m-c'-b'y).$$

We express the resulting ternary equation through a known integral; then our sum is transformed into the form

$$S\left(N,\begin{matrix}a & b & c\\ a' & b' & c'\end{matrix}\right)=\frac{1}{aa'}\sum_{\nu=0}^{a-1}\sum_{\nu'=0}^{a'-1}\exp\left[-2\pi i\left(\frac{c\nu}{a}+\frac{c'\nu'}{a'}\right)\right]\times$$
$$\times\int_0^1 S_1(\alpha)S_2(\alpha)T(\alpha)\exp(2\pi i\alpha A)\,d\alpha,$$

where

$$A=c'(u-ca'),$$

$$S_1(\alpha)=\sum_{n=1}^{\infty}\exp\left[-\frac{n}{N}+2\pi i\left(aa'+\frac{\nu}{a}\right)n\right]\Lambda(n),$$

$$S_2(\alpha)=\sum_{m=1}^{\infty}\exp\left[-\frac{m}{N}-2\pi i\left(\alpha a-\frac{\nu'}{a'}\right)m\right]\Lambda(m),$$

$$T(\alpha)=\sum_{y=1}^{\infty}\exp\left\{-\frac{|b|+|b'|}{N}y+2\pi i\left[\alpha(b'a-a'b)-\frac{b\nu}{a}-\frac{b'\nu'}{a'}\right]y\right\}$$

and, by assumption, $|b'a-a'b|\neq 0$.

The resulting integral

$$\int_0^1 S_1(\alpha)S_2(\alpha)T(\alpha)\exp(2\pi i\alpha A)\,d\alpha$$

is easily studied by means of the Hardy–Littlewood circle method, since the sum $T(\alpha)$ is complete and on the minor arcs is estimated trivially. After calculation we obtain

$$S\left(N, \begin{matrix} a & b & c \\ a' & b' & c' \end{matrix}\right) = \mathfrak{S}\left(\begin{matrix} a & b & c \\ a' & b' & c' \end{matrix}\right) N^2 + O\left(\frac{N^2}{\ln^C N}\right), \tag{2}$$

where $\mathfrak{S}$ is a singular series, greater than zero under the imposition of the natural congruence conditions on the coefficients, and $C > 0$ is an arbitrarily large constant. From (2) the answer to Hua Lo-keng' s question follows immediately.

Finally, let us note that this device is suitable for solving the more complicated system for a point with three prime coordinates:

$$\begin{aligned} a_1x + b_1y + c_1 &= p_1, \\ a_2x + b_2y + c_2 &= p_2, \\ a_3x + b_3y + c_3 &= p_3. \end{aligned} \tag{3}$$

After introducing the fivefold sum

$$S\left(N, \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}\right),$$

analogous to the triple sum (1), and after eliminating x and y twice from the system (3), we arrive at a ternary problem with primes:

$$Ap_1 + Bp_2 + Cp_3 = D, \tag{4}$$

where

$$A = a_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix},$$

$$B = a_1 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \qquad C = a_1 \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix},$$

$$D = \begin{vmatrix} a_3 & a_1 \\ c_3 & c_1 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} - \begin{vmatrix} a_2 & a_1 \\ c_2 & c_1 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}.$$

with the condition that $A, B, C \neq 0$, and with four congruences modulo

$$a_1, \quad a_2, \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}.$$

Treating equation (4), with additional congruences for prime numbers, by the classical circle method of Hardy–Littlewood–Linnik, we obtain, after computations,

$$S\left(N, \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\right) = \mathfrak{S}\left(\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\right) N^2 + O\left(\frac{N^2}{\ln^C N}\right),$$

where $\mathfrak{S}$ is a singular series, greater than zero under the natural solvability conditions for the coefficients of system (3). We note that, by complicating the arguments, one can similarly solve the problem only for positive x and y.

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CITED LITERATURE

1. Hua Lo-keng, *Additive Theory of Prime Numbers*, USSR Academy of Sciences Publishing House, 1947.

2. Yu. V. Linnik, *Matem. sbornik*, **19** (61), 3 (1946).

Note: Figure translations are in progress. See original paper for figures.