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Abstract

Full Text

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On the Question of Stability of Systems of Linear Integral Equations of Volterra Type

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Let us consider a system of Volterra integral equations

$$y(t) = \int_a^t G[t, s, y(s)]\, ds + f(t), \tag{1}$$

where $y(t) = \{y_1(t), \dots, y_n(t)\}$, $f(t) = \{f_1(t), \dots, f_n(t)\}$, $G(t, s, y) = \{G_1(t, s, y_1, \dots, y_n), \dots, G_n(t, s, y_1, \dots, y_n)\}$ are complex-valued vector functions continuous for $a \le s \le t < \infty$.

System (1) is called stable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that from

$$\sup_{t \ge a} \sum_{i=1}^n |f_i(t)| \le \delta$$

it follows that

$$\sup_{t \ge a} \sum_{i=1}^n |y_i(t)| \le \varepsilon.$$

Using theorems on integral inequalities (see, for example, ($^{1-5}$)), the question of stability of system (1) can be reduced to the investigation of the stability of a certain majorant system (^{3,4,6}). In the simplest case, a real linear system of Volterra integral equations may be taken as the majorant,

$$x(t) = \int_a^t K(t, s) x(s)\, ds + \varphi(t) \tag{2}$$

with a nonnegative matrix $K(t, s) = (K_{ij}(t, s))$. A number of works ($^{7-9}$) are devoted to the stability of systems of linear integral equations. The use of integral inequalities and of generalized norming (^{10}) makes it possible in a number of cases to obtain results more precise than those in the works cited above. We shall assume that in system (2) the matrix $K(t, s)$ is nonnegative for $a \le s \le t < \infty$.

Lemma (cf. (^{11})). *For the stability of system (2) it is necessary and sufficient that:*

1)

$$\sup_{t\geq a}\int_a^t \|K(t,s)\|\,ds < \infty.$$

2) *For some $T \geq a$ and $\alpha > 1$ there exists a continuous, bounded on $[T,\infty)$, essentially positive vector function $\gamma(t) = \{\gamma_i(t)\}$ $(\omega \geq \gamma_i(t) \geq \beta > 0,\ t \geq T,\ i = 1,\dots,n)$, such that*

$$\gamma(t) \geq \alpha \int_T^t K(t,s)\gamma(s)\,ds.$$

Everywhere below we shall assume condition 1) to be fulfilled. From the lemma there follows the following assertion.

Theorem 1. *If, for some $m \geq 1$, the matrix*

$$A_m = \lim_{T\to\infty}\lim_{t\to\infty}\int_T^t K_m(t,s)\,ds,$$

where $K_m(t,s)$ is the m-th iterated kernel, has characteristic numbers whose moduli are less than one, then system (2) is stable. Conversely, if the sys-

system (2) is stable, then there exists an m such that the characteristic numbers of the matrix A_m are, in modulus, less than one. Moreover, all the matrices

$$B_i = \lim_{T\to\infty}\lim_{t\to\infty}\int_T^t K_i(t,s)\,ds$$

also have this property.

If the characteristic numbers of the matrix A_1 are, in modulus, less than one, then the matrices A_i $(i \geq 2)$ also have this property. The converse is false. Therefore consideration of iterated kernels leads to more general criteria.

Corollary. If for every $T \geq T_0 \geq a$ there exists

$$\lim_{t\to\infty}\int_T^t K(t,s)\,ds,$$

then, for stability of system (2), it is necessary and sufficient that the characteristic numbers of the matrix

$$\lim_{T\to\infty} \sup_{t\geq T} \int_T^t K(t,s)\,ds$$

be, in modulus, less than one. In particular, for $K(t,s) = Q(t-s)$ system (2) is stable if and only if the characteristic numbers of the matrix

$$\int_0^\infty Q(s)\,ds$$

are, in modulus, less than one. Convenient criteria for the characteristic numbers of a nonnegative matrix to be, in modulus, less than one are given in (^{12}) (for example, p. 339).

Put $i_0 = 1 < i_1 < \dots < i_l = n+1$ and denote by $Q_{mk}(t,s)$ $(m,k = 1,\dots,l)$ the matrix with elements $K_{ij}(t,s)$, $i_{m-1} \leq i < i_m$, $i_{k-1} \leq j < i_k$. From the lemma and Theorem 1 it follows that

Theorem 2. If

$$\lim_{T\to\infty} \sup_{t\geq T} \int_T^t \|Q_{mk}(t,s)\|\,ds = 0$$

for $m < k$ (or for $m > k$), then system (2) is stable if and only if the equations

$$z(t) = \int_a^t Q_{kk}(t,s)z(s)\,ds + \psi(t) \qquad (k = 1,\dots,l)$$

are stable.

We note that Theorem 2 is valid also without the assumption of nonnegativity of the matrix $K(t,s)$.

Consider the system of differential equations

$$y' = A(t)y, \qquad A(t) = (a_{ij}(t)) \quad (t \geq 0,\ i,k = 1,\dots,n), \tag{3}$$

where $a_{ij}(t)$ are complex-valued functions continuous on $[0,\infty)$. Reducing, in one way or another (for example, by the method indicated in (^{5})), system (3) to an integral one and applying the theorems formulated above, one can obtain various stability criteria for system (3). As an example we give the following two propositions.

Theorem 3. Let $A(t) = A_1 + A_2(t)$, and let $Y(t)$ be the solution of the matrix equation $Y'(t) = A_1Y(t)$, $Y(0) = E$ (E is the identity matrix). If $\sup_{t\geq 0} \|Y(t)\| < \infty$ and the characteristic numbers of the matrix

$$\lim_{T\to\infty}\sup_{t\geq T}\int_T^t Y(t-s)A_2(s)\,ds\ *$$

are, in modulus, less than one, then all solutions of system (3) are bounded.

$$*\ \text{If } B=(b_{ij}),\ \text{ then } |B|=(|b_{ij}|).$$

From Theorem 2 it follows

Theorem 4. Suppose that there exist functions $b_i(t)$ such that

$$\sup_{t\geq 0}\int_0^t \operatorname{Re} b_i(s)\,ds<\infty,$$

$$\sup_{t\geq 0}\int_0^t |a_{ii}(s)-b_i(s)|\exp\left\{\int_s^t \operatorname{Re} a_{ii}(\tau)\,d\tau\right\}ds<\infty,$$

$$\sup_{t\geq 0}\int_0^t |a_{ij}(s)|\exp\left\{\int_s^t \operatorname{Re} b_i(\tau)\,d\tau\right\}ds<\infty \quad \text{for } i>j\ (i<j),$$

$$\lim_{T\to\infty}\sup_{t\geq T}\int_T^t |a_{ij}(s)|\exp\left\{\int_s^t \operatorname{Re} b_i(\tau)\,d\tau\right\}ds=0 \quad \text{for } i<j\ (i>j).$$

Then all solutions of system (3) are bounded.

Taking $b_i(t)=a_{ii}(t)$, we obtain a generalization of the theorems of N. I. Gavrilov (^{13}) and A. P. Moiseenko (^{14}).

In conclusion we note that Theorems 1 and 2 extend to multidimensional linear Volterra equations in the sense of A. N. Tikhonov (^{15}). In particular, these results are valid for equations of the form

$$x(t,s)=\int_a^t dy\int_G\sum_{j=1}^m K_j(t,s,y,z)\,x(y-g_j(y),z)\,dz+f(t,s)$$

$$(t,a\in R_\mu,\ \ s\in R_\nu,\ \ \mu,\nu\geq 1,\ \ m\hat{G}<\infty,\ \ g_j(y)\geq 0).$$

Theorems on integral inequalities for these equations were presented at the Izhevsk seminar by A. I. Logunov and N. V. Shklyaeva.

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Note: Figure translations are in progress. See original paper for figures.

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