# S. V. SMIRNOV

1963

SovietRxiv

## Abstract

## Full Text

# S. V. SMIRNOV

## NOMOGRAMS AND WEBS

*(Presented by Academician A. N. Kolmogorov on 6 X 1962)*

V. Blaschke [1] long ago noted the connection between nomograms and webs. In the author' s notes [2] the problem of nomographability under the assumption of sufficient smoothness was solved for functions of many variables. On the other hand, under the same assumption there are approaches to the solution of the problem of the rectifiability of webs by the methods of tensor differential geometry [3] or by the method of immersed manifolds [4], which, however, do not give a final solution of the problem. Meanwhile, this is the same problem of nomographability.

Here we indicate a method for obtaining necessary and sufficient conditions for the rectifiability of a web in the form of relations connecting the partial derivatives of the functions of the web. These relations are invariant in character.

Consider the web $T_{n+1}$:

$$\psi_j(\xi_1, \dots, \xi_n) = x_j, \qquad j = 0, 1, \dots, n, \tag{1}$$

formed by $n+1$ one-parameter families of hypersurfaces in $R_n$, an $n$-dimensional coordinate space $\xi_1, \dots, \xi_n$.

Let each of the variables $x_j$ range, respectively, over the numerical interval $\delta_j = [a_j, b_j]$, $j = 0, \dots, n$; denote by $G_i$ the $n$-dimensional parallelepiped—the topological product of all $\delta_j$ except only $\delta_i$:

$$G_i = \delta_0 \times \dots \times \delta_{i-1} \times \delta_{i+1} \times \dots \times \delta_n; \tag{2}$$

suppose that, writing down all equations (1) except only the one with number $i$, we obtain a sufficiently smooth homeomorphism mapping $G_i$ into $R_n$, and let this be true for $i = 0, \dots, n$. Denote the image of $G_i$ in $R_n$ by $\widetilde{G}_i$.

Consider $D = \bigcap_{i=0}^{n} \widetilde{G}_i$. If $D$ is a nonempty set, then, eliminating $\xi_1, \dots, \xi_n$ from equations (1), we find the "equation of the web" $T_{n+1}$:

[1] Reference 1.
[2] References 6-10.
[3] Reference 3.
[4] Reference 2.

$$F(x_0, x_1, \dots, x_n) = 0, \tag{3}$$

where, in view of the preceding, (3) can be solved with respect to each of the variables $x_0, x_1, \dots, x_n$. The intervals $\delta_j$ can be chosen in such a way that (3) is uniquely solvable and, moreover, any solution

$$x_i = \varphi_i(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \qquad i = 0, \dots, n, \tag{4}$$

is defined in the corresponding parallelepiped $G_i$. In this case $D$ is called a cell of the web $T_{n+1}$, and any of the equivalent equations (3), (4) is called an equation of this web.

In what follows it will only be required that all determinants of order $n$ formed from the Jacobi matrix $J = (\partial\psi_j/\partial\xi_i)$ do not vanish anywhere in the cell of the web. Obviously, in this case all partial derivatives $\partial\varphi_i/\partial x_k$ $(k \neq i)$, $i = 0, \dots, n$, are nonzero everywhere in $G_i$.

We also assume that all elements of the Jacobi matrix are differentiable in $D$ a sufficient number of times, i.e., possess a total differential of correspondingly high order. Below, in each case, it will be clear exactly what restrictions are imposed on the smoothness of the functions under consideration.

The variables $x_0, x_1, \dots, x_n$ are entirely equivalent in this investigation, and any one of them may be singled out when forming the corresponding equation (4). Below a privileged position is fixed for one of them; here it is $x_0$, and, in order to connect the exposition with the author' s preceding works ([6–10]), the notation $z$ is introduced instead of $x_0$. The equation of the web $T_{n+1}$ is written in the form

$$z = \varphi(x_1, \dots, x_n), \qquad (x_1, \dots, x_n) \in G, \tag{5}$$

where the index is omitted in $G_0$. This does not mean a violation of symmetry, since everything can be repeated for any of the variables. In particular, in this way certain simplifications may be obtained in the process of computing nomograms.

The problem of the rectifiability of the web ([1]) may be formulated as follows:

*Do there exist in $R_n$ one-parameter families of hyperplanes, i.e., a web $W_{n+1}$:*

$$\sum_{k=1}^{n} \xi_k f_{ik}(x_i) + 1 = 0, \qquad i = 0, \dots, n, \tag{6}$$

*such that the equation of the web $W_{n+1}$ is a nomographic representation ([9,10]) in the parallelepiped $G$ of equation (5) of the web $T_{n+1}$?*

In ([8,10]), as follows from the preceding formulation of the problem, necessary and sufficient conditions for the rectifiability of a web in differential form in the sense of Gronwall were obtained.

However, one can go further and indicate, as was done in the most important case in ([6]) (see also ([4,5])), necessary and sufficient conditions in the form of relations connecting the partial derivatives of the right-hand side of the web equation in the form (5).

The principal result from ([8,10]) can be given the following form.

**Definition.** We shall call a square matrix $C$ of order $n$ a **Gronwall matrix** if its elements $c_{ik}$, continuously differentiable in $G$, possess the following properties:

$$\text{a)} \quad c_{ii} \equiv 0, \qquad i = 1, \dots, n, \tag{7}$$

b) there exists a continuously differentiable square matrix $Z$ of order $n$, whose elements are connected with the elements of $C$ by the relations

$$\frac{\partial c_{si}}{\partial x_k} + (1 + \delta_{si}) z_{sk} = \frac{\partial c_{ki}}{\partial x_s} + (1 + \delta_{ki}) z_{ks}, \qquad i, k, s = 1, \dots, n, \tag{8}$$

holding in $G$, and, moreover, satisfy the differential equations

$$\frac{\partial z_{ik}}{\partial x_k} = c_{ki} z_{ik}, \qquad i, k = 1, \dots, n, \tag{9}$$

c) the "compatibility conditions" hold $(i, k, s = 1, \dots, n)$

$$2 \frac{\partial}{\partial x_s} (c_{si} - c_{sk}) = c_{si}^2 - c_{sk}^2, \qquad s \neq i, \ s \neq k. \tag{10}$$

The elements of the matrix $C$ are called **Gronwall functions**.

**Theorem 1.** *The function $z = \varphi(x_1, \dots, x_n)$, $(x_1, \dots, x_n) \in G$, is representable by a nomogram of aligned points if and only if in the parallelepiped G there exists a Gronwall matrix whose elements are connected with the partial derivatives of $\varphi(x_1, \dots, x_n)$ by the equalities*

$$\varphi_i \varphi_k (\varphi_i c_{ki} + \varphi_k c_{ik}) = \varphi_i^2 \varphi_{kk} - 2\varphi_i \varphi_k \varphi_{ik} + \varphi_k^2 \varphi_{ii}, \qquad 1 \leq i < k \leq n, \tag{11}$$

*which hold identically in the parallelepiped $G$.*

This theorem can be given the following equivalent form:

**Theorem 2.** The equation $z = \varphi(x_1, \dots, x_n)$ is nomographable if and only if this equation is nomographable for any $i \neq k$ as an equation relating the variables $x_i, x_k, z$, while all the remaining $x_j$ $(j \neq i,\ j \neq k)$ play the role of parameters, and the Gronwall functions and the adjoined functions (the elements of the matrix $Z$) of all such partial nomograms, as functions of the $n$ variables $x_1, \dots, x_n$, satisfy the connection conditions c), and, in addition, the following further conditions:

d) each of the differential forms

$$\sum_{j=1}^{n} z_{ij}\, dx_j, \qquad i = 1, \dots, n, \tag{12}$$

is, respectively, the partial differential of a function

$$v_i(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \tag{13}$$

with respect to the variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$;

e) each of the differential forms

$$\sum_{k=1}^{n} (1 + \delta_{ik})(c_{ki} + v_i)\, dx_k, \qquad i = 1, \dots, n, \tag{14}$$

is the complete differential of a function of all independent variables $x_1, \dots, x_n$.

**Proof.** Consider the system composed of the differential equations (9) and (11). This system decomposes into $n(n-1)/2$ subsystems, each of which, together with a pair of corresponding relations

$$3z_{ik} = 2\frac{\partial c_{ki}}{\partial x_i} + \frac{\partial c_{ik}}{\partial x_k},$$

$$3z_{ki} = \frac{\partial c_{ki}}{\partial x_i} + 2\frac{\partial c_{ik}}{\partial x_k}, \tag{15}$$

following from (8), gives the above-mentioned conditions for the nomographability of $z = \psi(x_1, \dots, x_n)$ as a function of the variables $x_i, x_k$, while the other variables $x_j$ $(j \neq i,\ j \neq k)$ play the role of parameters.

Conditions (12) and (14) are easily derived from (8); the converse is also true.

It is known ([4–6]) that, for the nomographability of $z = \varphi(x_1, \dots, x_n)$ as a function of only two variables $x_i, x_k$, the preceding conditions of Gronwall type may be replaced by identical relations between the partial derivatives of the right-hand side of (5) with respect to the variables $x_i, x_k$ up to and including order

8; moreover, these conditions, naturally, must be identically satisfied also with respect to the variables which in the given situation play the role of parameters.

It is also known ([7]) that, generally speaking, for any pair of indices $i, k$, the Gronwall functions $c_{ik}, c_{ki}$ and the adjoined functions $z_{ik}, z_{ki}$ are found as the solution of a system of algebraic equations with coefficients which, in turn, are polynomials in the partial derivatives of the right-hand side of (5) and are determined in an invariant manner.

From the preceding it follows:

**Theorem 3.** A necessary and sufficient condition for the representability of the equation $z = \varphi(x_1, \dots, x_n)$ by an $n$-dimensional nomogram of aligned points reduces to the collection of analogous conditions for partial nomograms with respect to the variables $x_i, x_k, z$ for all possible combinations $i \neq k;\ i, k = 1, \dots, n$, and of additional relations corresponding to the connection conditions (10), as well as relations corresponding to the conditions imposed on the differential forms (12) and (14).

If none of the functions, $i, k = 1, \dots, n$,

$$p_{ik} = \frac{\partial^2}{\partial x_i \partial x_k} \ln \frac{\varphi_i}{\varphi_k}, \qquad i \neq k, \tag{16}$$

vanishes in the parallelepiped $G$, then in this case all the relations between the partial derivatives of the right-hand side of (5) are obtained by substituting the roots of a certain system of polynomials in $c_{ik}, c_{ni}, z_{ik}, z_{ni}$ into the equations of relation and into (8).

If at least one of the functions (16) is identically zero in some subdomain $G$, then, alongside the algebraic relations described above, there arise differential relations imposed on the Gronwall functions and not reducible to algebraic ones. One can readily compile a complete catalogue of such cases, each of which is characterized by certain relations between the partials and derivatives of the right-hand side of (5).

All the foregoing can be expressed in the language of web theory, at least for a small cell of the web.

Received
5 X 1962

## CITED LITERATURE

1. W. Blaschke, *Einführung in die Geometrie der Waben*, 1955.

2. M. A. Bartoshevich, DAN, **124**, 970 (1959).

3. A. E. Liber, Proceedings of the Seminar on Vector and Tensor Analysis, issue IX (1956).

4. M. A. Kreines, N. D. Eisenstadt, DAN, **95**, 1137 (1954).

5. M. A. Kreines, N. D. Eisenstadt, Mat. sborn., **37** (79), 337 (1956).

6. S. V. Smirnov, DAN, **69**, 99 (1949).

7. S. V. Smirnov, DAN, **116**, 359 (1957).

8. S. V. Smirnov, Multidimensional nomograms, Abstracts of reports, scientific conference of 1959, Ivanovo Pedagogical Institute Press, 1959, p. 81.

9. S. V. Smirnov, DAN, **124**, 34 (1959).

10. S. V. Smirnov, DAN, **142**, 282 (1962).


*Note: Figure translations are in progress. See original paper for figures.*