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Abstract

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LAPLACE SEQUENCES OF GENERALIZED CONJUGATE ROZET NETS OF PROJECTIVE N-SPACE

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1. A **Rozet net** of three-dimensional projective space is defined as a conjugate net such that, for each of its points, the corresponding points of three successive Laplace transforms of the net in one direction and of one transform in the other lie in one plane (^{1}).

In the present paper generalized conjugate Rozet nets of N-dimensional projective space P_N are constructed, and the Laplace sequences built on them are considered.

2. Let

$$\dots,(A_0),(A_1),(A_2),\dots,(A_{N+1}),(A_{N+2}),\dots \tag{A}$$

be a Laplace sequence of the space P_N. Consider a moving frame $A_1A_2\dots A_{N+1}$, formed by $N+1$ linearly independent analytic points A_i $(i=1,2,\dots,N+1)$, whose corresponding geometric points describe the conjugate nets (A_i). The infinitesimal displacements of the vertices of the frame are determined by the equations

$$dA_i=\omega_i^jA_j \qquad (i,j=1,2,\dots,N+1), \tag{1}$$

where ω_i^j are linear differential forms satisfying the structure equations of the space P_N: $D\omega_i^j=[\omega_i^k\omega_k^j]$ $(i,j,k=1,\dots,N+1)$.

By the choice of the frame, the tangent planes to the surfaces (A_i) $(i=2,\dots,N)$ at the points A_i are the planes $A_{i-1}A_{iA_{i+1}}$; therefore

$$\omega_i^j=0 \qquad (i=2,\dots,N;\ j\neq i-1,i,i+1). \tag{2}$$

Now equations (1) for $i=N$ and $i=2$ take the form

$$dA_N=\omega_N^NA_N+\omega_N^{N-1}A_{N-1}+\omega_N^{N+1}A_{N+1}, \qquad dA_2=\omega_2^2A_2+\omega_2^3A_3+\omega_2^1A_1.$$

We shall assume the surfaces (A_N) and (A_2) to be nondegenerate. Then $\omega_N^{N+1}\neq 0$, $\omega_2^1\neq 0$. We take these two forms as the principal forms and denote them by

ω^1 and ω^2: $\omega_N^{N+1} = \omega^1$, $\omega_2^1 = \omega^2$. With our choice of the vertices of the frame, since the points A_i and A_{i+1} are foci of the ray $A_{iA_{i+1}}$ $(i = 1, 2, \dots, N)$, we have the following Pfaffian equations

$$\omega_i^{i-1} = a_i^{i+1}\omega^1 \ (i = 2, \dots, N-1); \qquad \omega_{N+1}^i = a_{N+1}^i \omega^1 \ (i = 1, 2, \dots, N-1);$$

$$\omega_{N+1}^N = a_{N+1}^N \omega^1 + b_{N+1}^N \omega^2. \tag{3}$$

Exterior differentiation of equations (3) gives

$$[\Delta a_i^{i+1}\omega^1] = 0, \qquad [\Delta a_{N+1}^i \omega^1] = 0, \qquad [\Delta a_{N+1}^N \omega^1] + [\Delta b_{N+1}^N \omega^2] = 0, \tag{4}$$

where

$$\Delta a_i^{i+1} = da_i^{i+1} + a_i^{i+1}(\omega_N^N - \omega_{N+1}^{N+1} + \omega_{i+1}^{i+1} - \omega_i^i),$$

$$\Delta a_{N+1}^1 = da_{N+1}^1 + a_{N+1}^1(\omega_N^N - 2\omega_{N+1}^{N+1} + \omega_1^1) + a_{N+1}^2\omega^2,$$

$$\Delta a_{N+1}^i = da_{N+1}^i + a_{N+1}^i(\omega_N^N - 2\omega_{N+1}^{N+1} + \omega_i^i) + (a_{N+1}^{i+1} b_{i+1}^i + a_{N+1}^1 b_1^i)\omega^2$$

$$(i = 2, \dots, N-1),$$

$$\Delta a_{N+1}^N = da_{N+1}^N + a_{N+1}^N(2\omega_N^N - 2\omega_{N+1}^{N+1}) + a_{N+1}^1 b_1^N \omega^2,$$

$$\Delta b_{N+1}^N = db_{N+1}^N + b_{N+1}^N(\omega_2^2 - \omega_1^1 - \omega_{N+1}^{N+1} + \omega_N^N).$$

In (3) and (4) only one half of all the Pfaffian and quadratic equations has been written down; the other half is obtained from those displayed by the substitution

$$\begin{pmatrix} 1 & 2 \dots [\frac{1}{2}(N+1)] & a & \omega^1 \\ N+1 & N \dots N+2-[\frac{1}{2}(N+1)] & b & \omega^2 \end{pmatrix}, \tag{5}$$

where the square brackets denote the integral part of a number. Since the surfaces (A_i) $(i = 2, \dots, N)$ are nondegenerate, we have

$$a_i^{i+1} \neq 0, \qquad b_{i+1}^i \neq 0 \qquad (i = 2, \dots, N-1). \tag{6}$$

3. For the proof of the main theorems we shall need the first prolongation of the system of equations (2), (3), (4). Let us expand equations (4):

$$\Delta a_i^{i+1} = a_{i1}^{i+1}\omega^1, \qquad \Delta a_{N+1}^i = a_{N+1,1}^i\omega^1,$$

$$\Delta a_{N+1}^N = a_{N+1,1}^N\omega^1 + a_{N+1,2}^N\omega^2, \qquad \Delta b_{N+1}^N = a_{N+1,2}^N\omega^1 + b_{N+1,2}^N\omega^2. \quad (7)$$

Exterior differentiation of equations (7) gives:

$$[\Delta a_{i1}^{i+1}\omega^1] = 0, \qquad [\Delta a_{N+1,1}^i\omega^1] = 0,$$

$$[\Delta a_{N+1,1}^N\omega^1] + [\Delta a_{N+1,2}^N\omega^2] = 0, \qquad [\Delta a_{N+1,2}^N\omega^1] + [\Delta b_{N+1,2}^N\omega^2] = 0, \quad (8)$$

where

$$\Delta a_{i1}^{i+1} = da_{i1}^{i+1} + a_{i1}^{i+1}(2\omega_N^N - 2\omega_{N+1}^{N+1} + \omega_{i+1}^{i+1} - \omega_i^i)+$$

$$+ a_i^{i+1}(2b_{N+1}^N - b_N^{N-1}a_{N-1}^N - a_{N+1}^1 b_1^{N+1} - 2b_{i+1}^i a_i^{i+1} + a_{i+1}^{i+2}b_{i+2}^{i+1} - a_{i-1}^i b_i^{i-1})\omega^2,$$

$$\Delta a_{N+1,1}^1 = da_{N+1,1}^1 + a_{N+1,1}^1(2\omega_N^N - 3\omega_{N+1}^{N+1} + \omega_1^1)+$$

$$+ \{a_{N+1}^1(3b_{N+1}^N - b_N^{N-1}a_{N-1}^N - 3a_{N+1}^1 b_1^{N+1} + a_1^2) + a_{N+1,1}^2\}\omega^2,$$

$$\Delta a_{N+1,1}^2 = da_{N+1,1}^2 + a_{N+1,1}^2(2\omega_N^N - 3\omega_{N+1}^{N+1} + \omega_2^2)+$$

$$+ \{a_{N+1}^2(3b_{N+1}^N - b_N^{N-1}a_{N-1}^N - 3a_{N+1}^1 b_1^{N+1} + a_2^3 b_3^2 + a_1^2)+$$

$$+ b_3^2 a_{N+1,1}^3 + b_1^2 a_{N+1,1}^1 - a_{N+1}^1 b_1^2\}\omega^2,$$

$$\Delta a_{N+1,1}^i = da_{N+1,1}^i + a_{N+1,1}^i(2\omega_N^N - 3\omega_{N+1}^{N+1} + \omega_i^i)+$$

$$+ \{a_{N+1}^i(3b_{N+1}^N - b_N^{N-1}a_{N-1}^N - 3a_{N+1}^1 b_1^{N+1} + a_i^{i+1}b_{i+1}^i + a_{i-1}^i b_i^{i-1})+$$

$$+ b^i_{i+1} a^{i+1}_{N+1,1} + b^{i-1}_i a^i_{N+1,1} - a^1_{N+1} b^{i-1}_1 a^i_{i-1}\}\omega^2 \quad (i = 3, \dots, N-1),$$

$$\Delta a^N_{N+1,1} = da^N_{N+1,1} + a^N_{N+1,1}(3\omega^N_N - 3\omega^{N+1}_{N+1}) + \{a^N_{N+1}(4b^N_{N+1} - 2b^{N-1}_N a^N_{N-1} -$$

$$- 3a^1_{N+1} b^{N+1}_1) + b^N_1 a^1_{N+1,1} - a^1_{N+1} b^{N-1}_1 a^N_{N-1}\}\omega^2,$$

$$\Delta a^N_{N+1,2} = da^N_{N+1,2} + a^N_{N+1,2}(\omega^2_2 - \omega^1_1 + 2\omega^N_N - 2\omega^{N+1}_{N+1}),$$

$$\Delta b^N_{N+1,2} = db^N_{N+1,2} + b^N_{N+1,2}(2\omega^2_2 - 2\omega^1_1 - \omega^{N+1}_{N+1} + \omega^N_N) +$$

$$+ b^N_{N+1}(2a^2_1 - a^3_2 b^2_3 - 2b^N_{N+1} + b^{N-1}_N a^N_{N-1})\omega^1.$$

4. We shall call the net (A_J) a **generalized conjugate Rozet net** and denote it by $\mathrm{P}^{(J)}_{k,N+1-k}$ (J is any integer, $0 \leq k \leq N+1$), if in the Laplace sequence $\{A\}$ generated by this net the corresponding point A_J and the points $A_{J-1}, A_{J-2}, \dots, A_{J-k}$ and $A_{J+1}, A_{J+2}, \dots, A_{J+N+1-k}$ lie in one plane. We shall consider generalized conjugate Rozet nets only for values of k satisfying the inequality $0 \leq k \leq [\frac{1}{2}(N+1)]$; for the remaining values of k the results are symmetric to those obtained and follow from them by means of the substitution (5).

5. Let (A_1) be a net $\mathrm{P}^{(1)}_{0,N+1}$, i.e., the point A_{N+2} lies in the hyperplane $A_2 A_3 \dots A_{N+1}$. The expansion of the point A_{N+2} in the vertices of the frame has the form:

$$A_{N+2} = b^N_{N+1} \sum_{i=1}^{N} a^i_{N+1} A_i - a^N_{N+1,2} A_{N+1}. \tag{9}$$

By virtue of the inequalities (6), in order to obtain a net $\mathrm{P}^{(1)}_{0,N+1}$, it is necessary to require $a^1_{N+1} = 0$. Equations (4), by virtue of the linear independence of the forms ω^1 and ω^2, then give $a^i_{N+1} = 0$ ($i = 2, \dots, N$). Investigation of the system shows that $q = 3N-2$, $s_1 = 3N-3$, $s_2 = 1$, i.e., it is in involution and determines nets $\mathrm{P}^{(1)}_{0,N+1}$ with the arbitrariness of one function of two arguments.

Since in this case $dA_{N+1} = \omega^{N+1}_{N+1} A_{N+1} + b^N_{N+1}\omega^2 A_N$, (A_N) is a developable surface whose edge of regression is the curve into which the surface (A_{N+1}) degenerates. Thus, for $k = 0$ we obtain a break in the sequence $\{A\}$ in one direction.

6. Let (A_{k+1}) be a net $\mathrm{P}^{(k+1)}_{k,N+1-k}$ $(1 \leq k \leq [\frac{1}{2}(N+1)])$, i.e., the point A_{N+2} lies in the hyperplane $A_1 A_2 \ldots A_{kA_{k+2}} A_{k+3} \ldots A_{N+1}$. Equality (9) shows that this will be the case if and only if

$$a^{k+1}_{N+1} = 0. \tag{10}$$

In this case the corresponding one of equations (4) gives:

$$a^{k+2}_{N+1} b^{k+1}_{k+2} + a^1_{N+1} b^{k+1}_1 = 0, \tag{11}$$

whence, by differentiating, we obtain

$$\begin{aligned} a^{k+2}_{N+1} \Delta b^{k+1}_{k+2} + b^{k+1}_{k+2} \Delta a^{k+2}_{N+1} + a^1_{N+1} \Delta b^{k+1}_1 + b^{k+1}_1 \Delta a^1_{N+1} = \{ & b^{k+1}_1 a^2_{N+1} \\ & + b^{k+1}_{k+2} (a^{k+3}_{N+1} b^{k+2}_{k+3} + a^1_{N+1} b^{k+2}_1)\} \omega^2 \\ & + a^1_{N+1} (b^k_1 a^{k+1}_k + b^{N+1}_1 a^{k+1}_{N+1}) \omega^1. \end{aligned} \tag{12}$$

(for $k = 1$ this equality is written in a somewhat modified form). Now: $q = 4N-4,\ s_1 = 4N-5,\ s_2 = 1$, i.e., the nets $\mathrm{P}^{(k+1)}_{k,N+1-k}$ $(1 \leq k \leq [\frac{1}{2}(N+1)])$ also depend on one function of two arguments.

7. Let us require that the net (A_k), adjacent to (A_{k+1}), also be of the same type $\mathrm{P}^{(k)}_{k,N+1-k}$. For this it is necessary and sufficient that the point A_0, whose expression in terms of the vertices of the frame is obtained from (9) by replacing (5), should be expanded in all vertices of the frame except A_k. It is easy to see that for $k = 1$ this condition will be

$$b^2_{11} = 0, \tag{13}$$

and for $1 < k \leq [\frac{1}{2}(N+1)]$

$$b^k_1 = 0. \tag{14}$$

Let us prove that in the case under consideration the net (A_{k+2}) is of the same type as the nets (A_k), (A_{k+1}), i.e., let us prove that the point A_{N+3} lies in the hyperplane $A_2 A_3 \ldots A_{kA_{k+1}} A_{k+3} A_{k+4} \ldots A_{N+1} A_{N+2}$. Since

$$dA_{N+2} = (\ldots) A_{N+2} + \omega^1 \left\{ b^N_{N+1} \sum_{i=2}^{N} \alpha^i_{N+1} A_i + (\ldots) A_{N+1} \right\} \pmod{\omega^2},$$

where

$$\alpha^i_{N+1} = a^i_{N+1,1} + a^{i-1}_{N+1} a^i_{i-1} - \frac{a^1_{N+1,1} a^i_{N+1}}{a^1_{N+1}} \quad (i = 2, \dots, N), \tag{15}$$

and by dots are denoted coefficients not of interest to us, the expression for A_{N+3} has the form

$$A_{N+3} = tA_{N+2} + \left\{ b^N_{N+1} \sum_{i=2}^{N} \alpha^i_{N+1} A_i + (\dots) A_{N+1} \right\}. \tag{16}$$

and therefore the condition of interest to us will be the condition

$$\alpha^{k+2}_{N+1} = 0. \tag{17}$$

Let us prove that this condition is identically satisfied. Indeed, equations (8), (12), taking (10), (13) or (14) into account, give

$$a^{k+2}_{N+1,1} b^{k+1}_{k+2} + b^{k+1}_1 a^1_{N+1,1} = 0. \tag{18}$$

Therefore

$$\alpha^{k+2}_{N+1} = a^{k+2}_{N+1,1} + a^{k+1}_{N+1} a^{k+2}_{k+1} - \frac{a^1_{N+1,1} a^{k+2}_{N+1}}{a^1_{N+1}} = \frac{1}{a^1_{N+1}} \left(a^{k+2}_{N+1,1} a^1_{N+1} - a^1_{N+1,1} a^{k+2}_{N+1} \right) =$$

$$= \frac{1}{a^1_{N+1} b^{k+1}_{k+2}} \left\{ a^1_{N+1} \left(a^{k+2}_{N+1,1} b^{k+1}_{k+2} + a^1_{N+1,1} b^{k+1}_1 \right) - a^1_{N+1,1} \left(a^{k+2}_{N+1} b^{k+1}_{k+2} + a^1_{N+1} b^{k+1}_1 \right) \right\} \equiv 0.$$

It is easy to see that from equalities (10), (17) there follows (14) for $k > 1$, and (13) for $k = 1$, i.e., if the nets (A_{k+1}), (A_{k+2}) are of type $\mathrm{P}_{k,N+1-k}$, then the net (A_k) is of the same type. Applying mathematical induction, we obtain that all nets of the sequence $\{A\}$ are also of the same type.

Thus, if two consecutive focal nets of the sequence $\{A\}$ belong to the class of nets $\mathrm{P}_{k,N+1-k}$, then all nets of this sequence are of the same type (a sequence of this type will be called a sequence $\mathrm{P}_{k,N+1-k}$).

8. Let us clarify the question of the existence of sequences $\mathrm{P}_{k,N+1-k}$. Here one must consider separately the cases $k = 1$, $k = 2$, $3 \le k \le [\frac{1}{2}(N+1)]$.

Investigation of the system (2)–(14) shows that sequences $\mathrm{P}_{k,N+1-k}$ depend on $4N - 6$ functions of one argument for $3 \le k \le [\frac{1}{2}(N+1)]$, on $4N - 5$ functions of one argument for $k = 2$, and on $4N - 4$ functions of one argument for $k = 1$.

9. The sequences $\mathrm{P}_{k,N+1-k}$ considered are a generalization of the sequences L considered by us earlier (^{2}). Thus, for $k = [\frac{1}{2}(N+1)]$ the sequence $\mathrm{P}_{k,N+1-k}$ coincides with the sequence $L^{2n+1}_{n,n}$, if $N = 2n+1$, and with the sequence $L^{2n}_{n-1,n}$, if $N = 2n$.

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Note: Figure translations are in progress. See original paper for figures.

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