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Abstract

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QUASI-INVARIANT KERNELS IN ANTAGONISTIC GAMES

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The most general results concerning the question of the existence of a solution of antagonistic games on the unit square with a bounded payoff function (for unexplained concepts and notation, see, for example, ([1])) were obtained by S. Karlin ([2]). In the present paper a description is given of a class on the unit square with an unbounded payoff function $K(x, y)$, for which the equality

$$\sup_F \inf_G \int_0^1 \int_0^1 K(x,y)\,dF(x)\,dG(y) = \inf_G \sup_F \int_0^1 \int_0^1 K(x,y)\,dF(x)\,dG(y), \quad (*)$$

holds, where F and G denote distributions on the interval $[0, 1]$.

Let $\varphi(x)$ be a one-to-one mapping of the interval $[0, 1]$ onto itself. For any distribution function F, by $\varphi(F)$ we denote the function defined by the equality

$$\varphi(F(x)) = \int_{\varphi^{-1}\{[0,x]\}} dF(x).$$

This function is also a distribution function.

Let $\varphi_\delta(x)$, $\psi_\delta(y)$ $(0 \le \delta \le \Delta)$ be two families of one-to-one transformations of the interval $[0, 1]$ onto itself. A kernel $K(x, y)$ is called **quasi-invariant** with respect to the families of transformations $\varphi_\delta(x)$, $\psi_\delta(y)$, if for every $\varepsilon > 0$ there exists a $\delta' > 0$ such that for all $\delta \in [0, \delta']$, x, and y,

$$|K(\varphi_\delta(x), y) - K(x, \psi_\delta(y))| < \varepsilon.$$

Consider a game Γ on the unit square, the payoff function $K(x, y)$ of which has the following properties:

1. The function $K(x, y)$ is measurable in each variable for every fixed value of the other.

2. The integrals $\int_0^1 K(x,y)\,dx$ and $\int_0^1 K(x,y)\,dy$ converge uniformly for all y and x, respectively.
3. For all $n > 1$ the games Γ_n with kernels $K_n(x,y)$ have solutions, where the functions $K_n(x,y)$ are defined as follows:

$$K_n(x,y) = \begin{cases} K(x,y), & |K(x,y)| < n, \\ n, & K(x,y) \geq n, \\ -n, & K(x,y) \leq -n. \end{cases}$$

4. There exist such families of transformations of the interval $[0,1]$ onto itself, φ_δ and ψ_δ, with respect to which the function $K(x,y)$ is quasi-invariant and for which, for all $\delta_1, \delta_2 \in [0,\Delta]$, $x \in [0,1]$, the inequalities

$$|\varphi_{\delta_1}(x) - \varphi_{\delta_2}(x)| > C|\delta_1 - \delta_2|,$$

$$|\psi_{\delta_1}(x) - \psi_{\delta_2}(x)| > C|\delta_1 - \delta_2|$$

hold.

Theorem 1. *If the function $K(x,y)$ satisfies conditions 1-4, then equality $(*)$ holds.*

Moreover, one can show that for every $\varepsilon > 0$ there exist N and $\delta' > 0$ such that, for all $n > N$, $\delta \in [0,\delta']$, the distribution functions

$$\overline{\varphi}_\delta(\widetilde{F}_n(x)) = \frac{1}{\delta}\int_0^\delta \varphi_\delta(\widetilde{F}_n(x))\,d\delta,$$

$$\overline{\psi}_\delta(\widetilde{G}_n(y)) = \frac{1}{\delta}\int_0^\delta \psi_\delta(\widetilde{G}_n(y))\,d\delta$$

($\widetilde{F}_n(x)$, $\widetilde{G}_n(y)$—optimal strategies of players I and II respectively in the games Γ_n) are ε-optimal strategies of the players in the game Γ.

From this theorem there immediately follow the facts established in (^{3}). A less trivial consequence is the following result.

Theorem 2. *Let the function $K(x,y)$ satisfy conditions 1-3 of Theorem 1 and, in addition, on the set of points of discontinuity be monotonically increasing in x and monotonically decreasing in y. Suppose, moreover, that the function $K(x,y)$ is quasi-invariant with respect to the following two families of transformations of subsets of the unit interval into the unit interval:*

$$\varphi_\delta^{(1)}(x) = \frac{x}{1-\delta}, \qquad x \in [0, 1-\delta]; \qquad \psi_\delta^{(1)}(y) = (1-\delta)y, \qquad y \in [0,1];$$

$$\varphi_\delta^{(2)}(x) = x - \delta, \qquad x \in [0,1]; \qquad \psi_\delta^{(2)}(y) = y + \delta, \qquad y \in [0, 1-\delta].$$

Then equality $(*)$ *holds.*

This theorem asserts the existence, for every $\varepsilon > 0$, of ε-equilibrium situations for games with kernels of the type

$$K(x,y) = \begin{cases} h_1(x,y)\ln(x-y), & x > y, \\ f(x), & x = y, \\ h_2(x,y)\ln(y-x), & x < y, \end{cases}$$

where h_1, h_2 are nonnegative functions continuous in both variables, defined respectively in the closed triangles $0 \le y \le x \le 1$ and $0 \le x \le y \le 1$; the functions h_1 and h_2 are monotonically increasing in x and monotonically decreasing in y, and the function $f(x)$ is bounded.

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CITED LITERATURE

[1] N. N. Vorob' ev, *Infinite Antagonistic Games*, 1963, p. 7.
[2] S. Karlin, ibid., p. 47.
[3] E. B. Yanovskaya, ibid., p. 77.

Note: Figure translations are in progress. See original paper for figures.