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Abstract

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THE ASSOCIATIVE SUPERENVELOPE OF A LIE ALGEBRA AND ITS INFINITE-DIMENSIONAL REPRESENTATIONS IN SPACES OF ANALYTIC GERMS

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Some features of the theory of infinite-dimensional representations of Lie groups allow one to hope that here a construction intermediate between a Lie algebra and a Lie group will be useful—the superenvelope of a Lie algebra. This construction, sufficiently meaningful for the natural formulation of the problem of its infinite-dimensional representations, at the same time does not entail restrictions on the choice of representations connected with the topological structure of the Lie group.

1°. Let G be an r-dimensional complex Lie algebra with basis $X_1, \dots, X_r$. For convenience of exposition the basis is normalized so that

$$\sum_k |c_{ij}^k| < 1 \quad (i, j = 1, \dots, r);$$

c_{ij}^k are the structure constants. As is known, the associative envelope $\mathfrak{A}(G)$ is defined as the ring of all polynomials in $X_1, \dots, X_r$, factored by the two-sided ideal generated by the polynomials

$$X_i X_j - X_j X_i - c_{ij}^k X_k$$

(with summation over k). Consider the totality of all formal series of the form

$$f = \sum_{s_1 \dots s_r} \frac{f_{s_1 \dots s_r}}{s_1! \cdots s_r!} ((X_1^{s_1} \cdots X_r^{s_r})), \tag{1}$$

where $s_1, \dots, s_r$ independently take the values $0, 1, 2, \dots$; the terms of the series are elements of the associative envelope, and the double round brackets mean that the product is averaged over all possible orders of its $s_1 + \dots + s_r$ factors $X_1, \dots, X_r$; $f_{s_1 \dots s_r}$ are arbitrary numerical coefficients, subject, however, for every $\varepsilon > 0$ to the condition

$$|f_{s_1 \dots s_r}| \le C_\varepsilon e^{\varepsilon s}, \qquad s = s_1 + \dots + s_r, \tag{2}$$

where $C_\varepsilon \geq 0$ is some constant (depending on ε and, of course, on f). In what follows, by $C_\varepsilon (= C_\varepsilon(f))$ we mean the least admissible value of C_ε.

Elements of the associative envelope are represented uniquely in the form of polynomials of the form (1) (the particular case where the series is finite).

In the totality of series (1) addition and multiplication by a number are naturally defined; in addition, we introduce an infinite system of norms

$$\|f\|_\varepsilon = C_\varepsilon(f) \qquad (\varepsilon > 0), \tag{3}$$

increasing as $\varepsilon \to 0$, and obtain a countably normed space $\mathfrak{F}$ in the sense of ([1]) (without changing the topology, one may, if desired, restrict oneself to a countable sequence of values $\varepsilon_i \to 0$). The system of norms (3) can easily be replaced by an equivalent system of Hilbert norms (in this case our space will be nuclear).

Now series of the form (1) will be absolutely convergent in each norm and, consequently, convergent in $\mathfrak{F}$. We introduce multiplication in $\mathfrak{F}$.

Lemma. If the elements $f_1, f_2 \in \mathfrak{A}(G)$ are written in the form of finite series–

the additive terms (1), then their product in $\mathfrak{A}(G)$ satisfies the inequality

$$|f_1 f_2|_\varepsilon \leq |f_1|_{\varepsilon_1} |f_2|_{\varepsilon_2} \frac{1}{1 - 2\sqrt{\Theta_\varepsilon}\dfrac{\varepsilon_1 + \varepsilon_2}{\varepsilon}} \tag{4}$$

under the condition that

$$2\sqrt{\Theta_\varepsilon}(\varepsilon_1 + \varepsilon_2) < \varepsilon < 2\pi, \qquad \Theta_\varepsilon = \sum_{i=0}^{\infty} |\alpha_i| \varepsilon^i, \quad \text{where } \sum_{i=0}^{\infty} \alpha_i x^i = \frac{x}{e^x - 1}.$$

It follows from estimate (4) that for two elements $f_1, f_2 \in \mathfrak{F}$ there exists in $\mathfrak{F}$ the limit of the product of their partial sums, which we take, by definition, to be the product $f_1 f_2$. Estimate (4) remains valid, and the product depends continuously on its factors. Associativity is easily derived. The results may be formulated as follows.

Theorem 1. *If in the associative envelope $\mathfrak{A}(G)$ of the Lie algebra G one introduces the infinite system of norms (3) and completes $\mathfrak{A}(G)$ with respect to this system of norms (i.e., takes the intersection of the closures with respect to each of the norms), then there arises a countably normed space $\mathfrak{F} = \mathfrak{F}(G)$, whose elements are represented in the form (1) and in which the operations of addition of elements, multiplication of them by a number, and multiplication of one by another may be defined, with the usual properties and with continuous dependence on the arguments; moreover, on $\mathfrak{A}(G)$ these operations coincide with the corresponding operations on $\mathfrak{A}(G)$.*

Thus, $\mathfrak{F}$ is a countably normed algebra over the field of complex numbers; we shall call it the **associative super-envelope of the Lie algebra** G.

2°. Let the Lie algebra G correspond to a local Lie group $\mathfrak{G}$, and let $u^1, \dots, u^r$ be canonical coordinates on $\mathfrak{G}$ in a neighborhood of the identity element e, corresponding to the basis $X_1, \dots, X_r$. Consider the linear space $\mathfrak{F}'$ of all analytic germs in a neighborhood of e:

$$\varphi(u^1, \dots, u^r) = \sum_{s_1,\dots,s_r} \frac{\varphi_{s_1 \dots s_r}}{s_1! \cdots s_r!} (u^1)^{s_1} \cdots (u^r)^{s_r}, \tag{5}$$

where $s_1, \dots, s_r = 0, 1, 2, \dots$, and $\varphi_{s_1 \dots s_r}$ are arbitrary numerical coefficients satisfying the condition

$$\left|\varphi_{s_1 \dots s_r}\right| \le C'_\varepsilon \varepsilon^{-s} s! \qquad (s = s_1 + \dots + s_r) \tag{6}$$

for at least one $\varepsilon > 0$ (and hence also for smaller $\varepsilon > 0$). In what follows, by $C'_\varepsilon (= C'_\varepsilon(\varphi))$ we mean the least C'_ε satisfying (6).

In the space $\mathfrak{F}'$ we introduce the infinite system of norms

$$\|\varphi\|'_\varepsilon = C'_\varepsilon, \tag{7}$$

decreasing as $\varepsilon \to 0$ (if, for a given ε, C'_ε does not exist, we set $\|\varphi\|'_\varepsilon = \infty$). We introduce a topology in $\mathfrak{F}'$: a set $M \in \mathfrak{F}'$ is called **closed** if, together with a sequence φ_i $(i = 1, 2, \dots)$, it also contains the element φ whenever

$$\|\varphi_i - \varphi\|'_\varepsilon \xrightarrow[i \to \infty]{} 0 \tag{8}$$

for at least one value of ε.

Theorem 2. *If $X_1, \dots, X_r$ are interpreted as the basic differential operators of left translations on $\mathfrak{G}$, and accordingly each element $f \in \mathfrak{F}$ as a "differential operator of infinite order" on $\mathfrak{G}$, defined by passage to the limit from the partial sum of the series (1), then the operator f will be defined for every $\varphi \in \mathfrak{F}'$, and $f\varphi \in \mathfrak{F}'$; moreover, the estimate holds*

$$\|f\varphi\|'_{\varepsilon_1} \le \|\varphi\|'_\varepsilon \, |f|_{\varepsilon_2} \left(1 - r\Theta_\varepsilon \varepsilon_2 \varepsilon_1^{-1} \ln^{-1} \frac{\varepsilon}{\varepsilon_1}\right) \tag{9}$$

for arbitrary $\varepsilon_1 < \varepsilon < 2\pi$ and for $\varepsilon_2 < \varepsilon \ln \dfrac{\tau}{\varepsilon_1} r^{-1} \Theta_\varepsilon^{-1}$.

Corollary. To each $f \in \mathfrak{F}$ there corresponds a continuous linear operator A_f on $\mathfrak{F}'$, $A_f \varphi = f\varphi$, which gives a natural representation of $\mathfrak{F}$ as an algebra of linear operators A_f on $\mathfrak{F}'$. This representation is continuous in the sense of the continuous dependence of $A_f \varphi$ on f for every fixed φ.

3°. Define the "scalar product" of $f \in \mathfrak{F}$ and $\varphi \in \mathfrak{F}'$ as the value of $f\varphi$ at the neutral element $e \in \mathfrak{G}$

$$(f, \varphi) = f\varphi\big|_e = \sum \frac{f_{s_1 \dots s_r} \varphi_{s_1 \dots s_r}}{s_1! \dots s_r!}. \tag{10}$$

Theorem 3. All continuous linear functionals in $\mathfrak{F}$ and in $\mathfrak{F}'$ have the form (10), with an arbitrarily fixed $\varphi \in \mathfrak{F}'$ in the first case and an arbitrarily fixed $f \in \mathfrak{F}$ in the second case.

Thus, the spaces $\mathfrak{F}$ and $\mathfrak{F}'$ may be regarded as mutually conjugate. The topology introduced by us in $\mathfrak{F}'$ is "superstrong" in comparison with the topologies introduced in ([1]).

By a plane $E \subset \mathfrak{F}$ and, analogously, $E' \subset \mathfrak{F}'$, we shall mean any closed linear subspace.

Theorem 4. Let us associate with each plane $E \subset \mathfrak{F}$ the plane $E' \subset \mathfrak{F}'$ consisting of all φ for which $(f, \varphi) = 0$ when $f \in E$; then different E' s correspond to different E'' s, and E' runs through the set of all planes in $\mathfrak{F}'$; the correspondence $E \to E'$ is thereby one-to-one, and E is determined by E' in the same way as E' is by E (the theorem is valid for any countably-Hilbert space $\mathfrak{F}$, provided that $\mathfrak{F}'$ is endowed with the "superstrong" topology).

Theorem 5. In order that the plane E' be invariant under the natural representation of the algebra $\mathfrak{F}$ on $\mathfrak{F}'$ $(f \to A_f)$, it is necessary and sufficient that the corresponding plane $E \subset \mathfrak{F}$ be a right ideal in $\mathfrak{F}$; in particular, for the irreducibility of E' it is necessary and sufficient that the right ideal E be maximal. If the elements $f \in \mathfrak{F}$ are turned into operators $\tilde{f}$, replacing in (1) $X_1, \dots, X_r$ by the basic operators of right translations (and not left translations, as up to now), then the plane E becomes, in the operator sense, a left ideal, and the operators $\tilde{f} \in E$ annihilate $\varphi \in E'$ identically (and not only at e); the one-to-one correspondence between the ideals E and the invariant planes E' can also be characterized in this way.

One can establish a connection between the constructions described and (strongly) continuous representations $g \to T_g$ of real Lie groups $\mathfrak{G}$, bounded by linear operators T_g in a Banach space B (T_{gx}, $x \in B$, is continuous in the argument $g \in \mathfrak{G}$).

The representation T admits an everywhere dense invariant subspace B^* in B of analytic vectors ([2]) with a system of decreasing norms (analogously to $\mathfrak{F}'$); here the algebra $\mathfrak{F}$ obtains a natural representation in B^* (analogous to the representation in $\mathfrak{F}'$). Moreover, B^* can be modeled in the form of a (generally speaking, non-closed) subspace $E^* \subset \mathfrak{F}'$, by assigning to each $x \in B^*$ the function, analytic on $\mathfrak{G}$,

$$x \to \varphi_x(g) = (\xi, T_{gx}) \tag{11}$$

and hence the analytic germ in a neighborhood of $e \in \mathfrak{G}$; here ξ is a functional in B; if the hyperplane $(\xi, x) = 0$ contains no invariant planes, then the correspondence (11) is one-to-one.

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Note: Figure translations are in progress. See original paper for figures.