MATHEMATICS

1963

SovietRxiv

Abstract

Full Text

MATHEMATICS

WANG SHEN-WANG

SOME REMARKS ON SOLUTIONS OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS

(Presented by Academician A. N. Kolmogorov on February 8, 1963)

In the present paper, solutions of certain nonlinear differential equations are studied by means of variational methods.

1. Several remarks concerning convex functionals. Let E be a Banach space, E_0 a linear set in E, with $\overline{E_0} = E$. We shall say that a functional $\varphi(u)$, defined on E_0, is **convex** (see ([1])) if

$$\varphi(tu + (1-t)v) \leq t\varphi(u) + (1-t)\varphi(v)$$

for $0 \leq t \leq 1;\ u, v \in E_0$.

Suppose that the operator $P(u)$ acts from E_0 into E^*, where E^* is the space conjugate to E. We shall say that the operator $P(u)$ is **nondecreasing** (respectively, **increasing**) (see ([1])) if, for any $u, h \in E_0,\ h \neq 0$, the inequality

$$(P(u+h) - P(u), h) \geq 0 \quad (> 0)$$

is fulfilled.

Let l be the segment

$$u = u_0 + t(u_1 - u_0), \quad 0 \leq t \leq 1,$$

lying in E_0. It is obvious that

$$\varphi(t) = (P(u_0 + t(u_1 - u_0)),\, u_1 - u_0)$$

is nondecreasing (respectively, increasing). Consequently, the integral

$$\int_0^1 (P(u_0 + t(u_1 - u_0)),\, u_1 - u_0)\, dt,$$

denoted by

$$\int_l (P(x), dx),$$

exists in the Riemann sense. If l is a polygonal line in E_0 with initial point u_0 and endpoint u_1, then

$$\int_l (P(x), dx)$$

is, by definition, the sum of the integrals of $P(u)$ over all segments of the polygonal line l. In the case where

$$\int_l (P(x), dx)$$

depends only on u_0 and u_1, we shall denote it by

$$\int_{u_0}^{u_1} (P(x), dx).$$

We shall say that the operator $P(u)$ is **potential** on E_0 if one can specify a functional $\varphi(u)$, defined on E_0, such that

$$\lim_{t\to 0} \frac{\varphi(u+th) - \varphi(u)}{t} = (P(u), h) \quad (u, h \in E_0).$$

Lemma 1. *Let the functional $\varphi(u)$, defined on E_0, be convex. Let it be differentiable in the sense of Gâteaux and let the Gâteaux differential $D(\varphi(u), h)$ be continuous in h. Then there exists a Gâteaux derivative $\varphi'(u)$. The operator $\varphi'(u)$ is a nondecreasing potential operator and*

$$\varphi(u) = \varphi(0) + \int_0^u (\varphi'(x), dx). \tag{1}$$

Proof. For the existence of $\varphi'(u)$ it is enough to show that $D(\varphi(u), h)$ is additive in h. Letting $t \to +0$ in the inequality

$$[\varphi(u + t(h+k)) - \varphi(u)]/t \le [\varphi(u+2th) - \varphi(u)]/2t + [\varphi(u+2tk) - \varphi(u)]/2t,$$

we obtain that

$$D(\varphi(u), h+k) \le D(\varphi(u), h) + D(\varphi(u), k),$$

whence in

by virtue of the homogeneity of $D(\varphi(u), h)$ with respect to h, it follows immediately that $D(\varphi(u), h+k) = D(\varphi(u), h) + D(\varphi(u), k)$.

From the convexity of the functional $\varphi(u)$ it follows that $\varphi'(u)$ is a nondecreasing operator. Let $u_0, u_1 \in E_0$ be arbitrary, and let l be the segment joining u_0 and u_1. Then $\eta(t) = \varphi(u_0 + t(u_1 - u_0))$ is absolutely continuous and $\eta'(t) = (\varphi'(u_0 + t(u_1 - u_0)), u_1 - u_0)$. Therefore

$$\int_l (\varphi'(x), dx) = \int_0^1 \eta'(t)\,dt = \eta(1) - \eta(0) = \varphi(u_1) - \varphi(u_0). \tag{2}$$

From (2) the validity of (1) follows directly. The lemma is proved.

The converse to Lemma 1 is

Lemma 2. *Let $P(u)$ be a nondecreasing potential operator. Then the functional*

$$\varphi(u) = c + \int_0^u (P(x), dx) \tag{3}$$

will be a convex functional and $\varphi'(u) = P(u)$.

Let us also note that, for a nondecreasing operator to be potential, it is necessary and sufficient that

$$\oint (P(x), dx) = 0$$

for any closed polygonal line in E_0.

2. On solutions of functional equations. Consider the equation

$$P(u) = f, \tag{4}$$

where $P(u)$ is an operator acting from E_0 into E^*, $f \in E^*$. A generalization of the theorem from [(2)] is

Theorem. *Let $P(u)$ be an increasing potential operator defined on E_0, $P(0) = 0$. Suppose that a solution of equation (4) exists. Then: 1) it is unique, 2) it is a point of minimum of the functional*

$$\Phi(u) = \int_0^u (P(x), dx) - (f, u), \tag{5}$$

and conversely: 3) a point of minimum $u_0 \in E_0$ of the functional (9) is a solution of equation (4). If, in addition, one assumes that

$$(P(u+h) - P(u), h) \geq \delta^2 \|h\|^\nu \quad (\delta > 0,\ \nu > 1), \tag{6}$$

then: 4) the functional $\Phi(u)$ is bounded from below and every minimizing sequence $\{u_n\}$ tends to the unique point $u_0 \in E$ in the metric of the space E.

Proof. Assertions 1), 3) are obvious. Let u_0 be a solution of equation (4). We have

$$\Phi(u_0+h) - \Phi(u_0) = \int_0^1 (P(u_0+th), h)\, dt - (f, h) = \int_0^1 (P(u_0+th) - P(u_0), h)\, dt > 0,$$

and assertion 2) is proved.

From condition (6) it follows that

$$\Phi(u) = \int_0^1 (P(tu), u)\, dt - (f, u) \geq$$

$$\geq \int_0^1 \frac{1}{t}(P(tu), tu)\,dt - (f, u) \geq \frac{\delta^2}{\nu}\|u\|^\nu - \|f\|\,\|u\|.$$

Since $\nu > 1$, the functional $\Phi(u)$ is bounded from below. Also, using condition (6)

we obtain that

$$\frac{1}{2}\Phi(u_n) + \frac{1}{2}\Phi(u_m) - \Phi\left(\frac{u_n + u_m}{2}\right) =$$
$$= \frac{1}{2}\left\{\Phi(u_n) - \Phi\left(\frac{u_n + u_m}{2}\right)\right\} + \frac{1}{2}\left\{\Phi(u_m) - \Phi\left(\frac{u_n + u_m}{2}\right)\right\} =$$
$$= -\frac{1}{2}\int_0^1 \left(P\left(u_n + \frac{1}{2}t(u_m - u_n)\right), \frac{u_m - u_n}{2}\right)\,dt$$
$$+ \frac{1}{2}\int_0^1 \left(P\left(u_n + \frac{1}{2}(1+t)(u_m - u_n)\right), \frac{u_m - u_n}{2}\right)\,dt \geq \left(\frac{1}{2}\right)^\nu \frac{\delta^2}{\nu}\|u_n - u_m\|^\nu, \tag{7}$$

whence follows the validity of the second part of assertion 4). The theorem is proved.

If the element $u_0 \in E$, which is the limit of the minimizing sequence $\{u_n\}$, is called a **generalized** solution of equation (4), then we arrive at the following corollary.

Corollary. *If $P(u)$ is a potential operator satisfying inequality (6), then for every $f \in E^*$ there exists a unique generalized solution of equation (4).*

3. Application. In the works of L. M. Kachanov, A. I. Koshelev, and A. Langenbach ([2–5]), the generalized solution of the equation

$$P_{Tu} \equiv -\frac{\partial}{\partial x}\left\{f[T^2(u)]\frac{\partial u}{\partial x}\right\} - \frac{\partial}{\partial y}\left\{f[T^2(u)]\frac{\partial u}{\partial y}\right\} = \omega \tag{8}$$

with the boundary condition

$$u|_S = 0, \tag{9}$$

was studied, where $T^2(u) = (\operatorname{grad} u)^2$, $u(x, y)$ is defined in a bounded domain Ω with piecewise-smooth boundary S. Let $E = L^2(\Omega)$, and let E_0 be the linear set of functions from the Hölder space E^2_α satisfying condition (9). With the help of Theorem 2 we can prove that there exists a unique generalized solution of equation (8), if the conditions

$$f(T^2) \geq f_0 > 0, \qquad f(T^2) + \alpha f'(T^2)T^2 \geq \chi > 0 \tag{10}$$

are fulfilled, and if $f(\xi)$ is continuously differentiable.*

Setting $I(u) = \int_0^u f(\xi)\,d\xi,\ P = P_T$, we obtain

$$\int_0^1 (P(u+th),h)\,dt = -\int_0^1 \left\{\int_\Omega \left\{\frac{\partial}{\partial x}\left[f(T^2(u+th))\frac{\partial(u+th)}{\partial x}\right]\right.\right.$$

$$\left.\left.+\frac{\partial}{\partial y}\left[f(T^2(u+th))\frac{\partial(u+th)}{\partial y}\right]\right\} h\,d\Omega\right\} dt = \frac{1}{2}\int_\Omega \left\{\int_0^1 f(T^2(u+th))\,d(T^2(u+th))\right\} d\Omega,$$

whence it follows that $P(u)$ is a potential operator on E_0. Since

$$\frac{d}{dt}(P(u+th),h) = \frac{d}{dt}\int_\Omega f(T^2(u+th))\left[\frac{\partial(u+th)}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial(u+th)}{\partial y}\frac{\partial h}{\partial y}\right] d\Omega =$$

$$= \int_\Omega \{f(T^2(u+th)) + \alpha f'(T^2(u+th))T^2(u+th)\}\left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right] d\Omega,$$

* A. Langenbach in his works ([2,3]) assumed that $f(\xi)$ is twice continuously differentiable.

then from conditions (10) and Friedrichs' inequality we obtain that $\frac{d}{dt}(P(u+th),h) \geq \gamma\|h\|^2$. Therefore

$$(P(u+h) - P(u),h) \geq \gamma\|h\|^2, \tag{11}$$

where $\gamma > 0$ is a constant. Thus all the conditions of Theorem 2 are satisfied.

In the case considered above we can prove that the generalized solution is in fact a weak solution. For this purpose it is enough to note that in proving inequality (11) one can obtain the stronger inequality

$$(P(u+h) - P(u),h) \geq \alpha\,\frac{d}{dt}(P(th),h)\Big|_{t=0} =$$

$$= \alpha f(0)\int_\Omega \left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right] d\Omega \geq \beta\|h\|^\nu \quad (\alpha,\beta > 0).$$

If we consider the scalar product

$$[u, v] = \frac{d}{dt}(P(tu), v)\Big|_{t=0} = f(0)\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) d\Omega,$$

then we obtain a new Hilbert space H_0, with $E_0 \subset H_0 \subset L^2(\Omega)$. The minimizing sequence $\{u_n\}$ converges in the metric of H_0 to the minimum point of the functional $\Phi(u)$, extended to H_0.

In a similar way one proves the validity of assertions B and C.

Appendix B. If the conditions

$$g(H^2) \geq c > 0, \qquad g(H^2) + \alpha g'(H^2)H^2 \geq \chi > 0 \tag{12}$$

are satisfied and $g(\xi)$ is twice continuously differentiable, then there exists a unique generalized solution of the equation*

$$P_{Hu} \equiv \frac{\partial^2}{\partial x^2}\left\{g(H^2(u))\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{\alpha}\frac{\partial^2 u}{\partial y^2}\right)\right\} +$$

$$+\frac{\partial^2}{\partial y^2}\left\{g(H^2(u))\left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{\alpha}\frac{\partial^2 u}{\partial x^2}\right)\right\} + \frac{\partial^2}{\partial x \partial y}\left\{g(H^2(u))\frac{\partial^2 u}{\partial x \partial y}\right\} = \omega, \tag{13}$$

satisfying the boundary conditions

$$u|_S = 0, \qquad \frac{\partial u}{\partial n}\Big|_S = 0. \tag{14}$$

Further, the generalized solution is also weak.

Appendix C. If the conditions

$$f(T^2) \geq f_0 > 0, \qquad f(T^2) + \alpha f'(T^2)T^2 \geq 0 \tag{15}$$

are satisfied and $f(\xi)$ is twice continuously differentiable, then there exists a unique generalized solution of the equation**

$$P_{Ru} \equiv \frac{\partial^2}{\partial x^2}\left\{f(R^2(u))\frac{\partial^2 u}{\partial x^2}\right\} + 2\frac{\partial^2}{\partial x \partial y}\left\{f(R^2(u))\frac{\partial^2 u}{\partial x \partial y}\right\} +$$

$$+\frac{\partial^2}{\partial y^2}\left\{f(R^2(u))\frac{\partial^2 u}{\partial y^2}\right\} + k\Delta^2 u - \frac{1}{3}\Delta\{f(R^2(u))\Delta u\} = 0, \tag{16}$$

satisfying the boundary conditions

$$u|_S = \varphi(s), \qquad du/dn|_S = \psi(s). \tag{17}$$

Further, the generalized solution is also a weak solution.

Mathematics Faculty
Nanking University
Nanking, PRC

Received
12 VII 1962

REFERENCES

1. R. I. Kachurovskii, *UMN*, **15**, 4 (1960).
2. A. Langenbach, *DAN*, **121**, No. 2 (1958).
3. A. Langenbach, *Vestn. LGU*, No. 1 (1961).
4. L. M. Kachanov, *Mechanics of Plastic Media*, Moscow–Leningrad, 1948.
5. A. I. Koshelev, *DAN*, **99**, No. 3 (1954).
6. M. M. Vainberg, *Variational Methods for the Investigation of Nonlinear Operators*, Moscow, 1956.
7. M. M. Vainberg, R. I. Kachurovskii, *DAN*, **129**, No. 4 (1959).

* $H^2(u) = (\partial^2 u/\partial x^2)^2 + (\partial^2 u/\partial y^2)^2 + (\partial^2 u/\partial x^2)(\partial^2 u/\partial y^2) + (\partial^2 u/\partial x \partial y)^2$.

** $R^2(u) = (\partial^2 u/\partial x \partial y)^2 + \frac{1}{3}\left[(\partial^2 u/\partial x^2)^2 + (\partial^2 u/\partial y^2)^2 + (\partial^2 u/\partial x^2)(\partial^2 u/\partial y^2)\right]$.

Note: Figure translations are in progress. See original paper for figures.