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Abstract

Full Text

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ON THE THEORY OF LINEAR POLYNOMIAL OPERATIONS

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1°. Denote by C the space of all continuous functions $f(x)$ defined on the segment $[-1, 1]$, with norm

$$\|f\| = \max_{-1 \le x \le 1} |f(x)|.$$

Denote by Π_n the set of all algebraic polynomials of degree $\le n$. Denote by $\mathfrak{M}_n$ the set of all linear operations V_n from C into C that map C into Π_n. The following theorem is known.

Theorem 1. In order that the sequence $\{V_n(f)\}_{n=1}^{\infty}$, $V_n \in \mathfrak{M}_n$, satisfy, for every $f \in C$, the relation

$$\|V_n(f) - f\| \to 0, \qquad n \to \infty; \tag{1}$$

it is necessary and sufficient that the following conditions hold: 1) for every polynomial relation (1) is fulfilled; 2) the norms of the operators V_n are bounded in the aggregate

$$\|V_n\| \le C, \qquad n = 1, 2, \dots,$$

where the constant C does not depend on n.

Verification of condition 2) is usually associated with great difficulties. Therefore one tries to replace it by other conditions. Thus, for example, if V_n is a method of summation of a Fourier–Chebyshev series,

$$V_n(f) = V_n(f, x, \lambda) = \frac{a_0 \lambda_0^{(n)}}{2} + \sum_{k=1}^{n} a_k \lambda_k^{(n)} T_k(x), \tag{2}$$

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(\cos t) \cos kt \, dt,$$

then condition 2) is replaced by the condition ([1,2])

$$\int_0^{\pi} \left| \frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} \cos kt \right| dt \le C, \qquad n = 1, 2, \dots .$$

Not every operator from $\mathfrak{M}_n$ is an operator of the form (2). Therefore the question arises of replacing, in the case of arbitrary linear polynomial operations

V_n, condition 2) by other conditions which, together with condition 1), are necessary and sufficient for relation (1) to hold for every $f \in C$. In the present note this question is studied in the nonperiodic case.

2°. Denote by $C^\wedge$ the space consisting of all even 2π-periodic functions $\varphi(\theta)$ with norm

$$\|\varphi\| = \max_{0\le\theta\le\pi} |\varphi(\theta)|.$$

Between the spaces $f \in C$ and $\varphi \in C^\wedge$ one can establish a one-to-one correspondence according to the formulas

$$\varphi(\theta) = f(\cos\theta), \qquad 0 \le \theta \le \pi;$$

$$f(x) = \varphi(\arccos x), \qquad -1 \le x \le 1,$$

where $f \in C$ and $\varphi \in C^\wedge$. It is clear that $\|f\| = \|\varphi\|$. As is known (1), φ is called the function induced by the function f. To preserve the connection in notation between the function itself and the induced function, we shall use

will be denoted by the sign $\hat{}$. Thus,

$$\hat{f}(\theta) = f(\cos\theta), \qquad f(x) = \hat{f}(\arccos x).$$

Following Faber ([3]) and Fejér ([4]), we define the shift T_h of a function $\varphi \in C^{\hat{}}$ by h according to the formula

$$T_h\varphi = T_h^\theta\varphi = \frac{\varphi(\theta+h)+\varphi(\theta-h)}{2}. \tag{3}$$

The operator (3) has the following obvious properties: 1) from $\varphi \in C^{\hat{}}$ it follows that, for every $-\infty < h < \infty$, $T_h\varphi \in C^{\hat{}}$; 2) $\|T_h\varphi\| \le \|\varphi\|$; 3) if φ is a cosine polynomial of order n, then $T_h^\theta\varphi$ is also a cosine polynomial of order n, both with respect to θ and with respect to h.

By the shift of a function $f \in C$ we mean the shift of the induced function. Thus,

$$T_h f = T_h^\theta f = \frac{\hat{f}(\theta+h)+\hat{f}(\theta-h)}{2}.$$

3°. Let $V_n \in \mathfrak{M}_n$. Introduce the operator

$$\widetilde{V}_n^{\hat{}}(f,\theta) = \frac{1}{\pi}\int_0^{2\pi} T_h^\theta V_n^{\hat{}}(T_h^\theta f)\,dh, \tag{4}$$

where h is considered as a parameter, and θ as the independent variable. The operator $\widetilde{V}_n$ has the following properties: 1) if $V_n \in \mathfrak{M}_n$, then $\widetilde{V}_n \in \mathfrak{M}_n$; 2) $\|\widetilde{V}_n\| \le 2\|V_n\|$.

Theorem 2. Let $V_n \in \mathfrak{M}_n$; then, for every $f \in C$, the equality

$$\widetilde{V}_n^{\hat{}}(\hat{f}, \theta) = \frac{1}{\pi} \int_0^{\pi} \hat{f}(h) \widetilde{V}_n^{\hat{}}(T_h^{\theta} D_n) \, dh, \tag{5}$$

holds, where $D_n(t)$ is the Dirichlet kernel of order n.

We indicate the proof of this theorem. It is known that the partial sum of order n of the Fourier series of the function $\hat{f}$ can be represented in the form

$$S_n(\hat{f}, \theta) = \frac{1}{\pi} \int_0^{\pi} \hat{f}(h) T_h^{\theta} D_n \, dh. \tag{6}$$

Since $\widetilde{V}_n \in \mathfrak{M}_n$, according to (5) we have

$$\widetilde{V}_n^{\hat{}}(\hat{f}) = \widetilde{V}_n^{\hat{}}(S_n(\hat{f})). \tag{7}$$

It follows from (6) and (7) that

$$\widetilde{V}_n^{\hat{}}(\hat{f}, \theta) = \frac{1}{\pi} \widetilde{V}_n^{\hat{}} \left(\int_0^{\pi} \hat{f}(h) T_h^{\theta} D_n \, dh \right). \tag{8}$$

It is easy to verify the validity of interchanging the operator and the integral. Therefore (5) follows from (8).

Theorem 3. Let $V_n \in \mathfrak{M}_n$; then

$$\|V_n\| \geq \frac{L_n}{2}, \qquad L_n = \max_{0 \leq \theta \leq \pi} \frac{1}{\pi} \int_0^{\pi} \left| \widetilde{V}_n^{\hat{}}(T_h^{\theta} D_n) \right| dh. \tag{9}$$

Proof. According to Theorem 2, $\widetilde{V}^{\hat{}}(\hat{f}, \theta)$ is an integral operator from $C^{\hat{}}$ into $C^{\hat{}}$ with kernel $\widetilde{V}^{\hat{}}(T_h^{\theta} D_n)$. Using a general theorem of functional analysis concerning integral operators, we obtain that

$$\|\widetilde{V}_n^{\hat{}}\| = L_n. \tag{10}$$

Since $\|V_n\| \geqslant \|\widetilde{V}_n^{\wedge}\|/2$, (9) follows from (10).

From Theorem 3 there follows

Theorem 4. *In order that the sequence* $\{V_n(f)\}_{n=1}^{\infty}$, $V_n \in \mathfrak{M}_n$, *satisfy relation (1) for every* $f \in C$, *it is necessary that relation (1) hold for every monomial and that*

$$L_n \leqslant C, \qquad n = 1, 2, \dots,$$

where the constant C does not depend on n.

It is curious that the necessary conditions indicated in Theorem 4 are close to sufficient conditions. This is seen from Theorem 5.

Theorem 5. *Let the sequence $\{V_n(f)\}_{n=1}^{\infty}$, $\mathfrak{M} \ni V_n$, be such that for every monomial relation (1) holds, and let*

$$L_n \leqslant C, \qquad n = 1, 2, \dots, \tag{11}$$

where the constant C does not depend on n. Then for every $f \in C$ the equality holds

$$\lim_{n\to\infty} \left\| \widetilde{V}_n^{\wedge}(f^{\wedge}) - f^{\wedge} - \frac{1}{\pi}\int_0^{\pi} f^{\wedge}(t)\,dt \right\| = 0. \tag{12}$$

Proof. From (10) and (11) it follows that the norms of the operators $\widetilde{V}_n^{\wedge}$ are uniformly bounded. Therefore, by Theorem 1, it remains only to prove that equality (12) holds for any function $T_m^{\wedge}(\theta) = \cos m\theta$. Let us first verify that equality (12) holds for $\cos m\theta$, where $m > 0$.

From the definition of $\widetilde{V}_n^{\wedge}$ we have

$$\widetilde{V}_n^{\wedge}(\cos m\theta) = \frac{1}{\pi}\int_0^{2\pi} T_h^{\theta} V_n^{\wedge}(\cos m\theta)\cos mh\,dh. \tag{13}$$

It is easy to see that

$$\cos m\theta = \frac{1}{\pi}\int_0^{2\pi} T_h^{\theta}\cos m\theta\cos mh\,dh. \tag{14}$$

From equalities (13) and (14) we conclude that

$$\left\| \widetilde{V}_n^{\wedge}(\cos m\theta) - \cos m\theta \right\| \leqslant 2\left\| V_n^{\wedge}(\cos m\theta) - \cos m\theta \right\|.$$

Since, by assumption, the right-hand side tends to zero as $n \to \infty$, we have

$$\lim_{n\to\infty} \left\| \widetilde{V}_n^{\wedge}(\cos m\theta) - \cos m\theta \right\| = 0,$$

and, for $f^{\wedge}(\theta) = \cos m\theta$, $m > 0$, this equality is equivalent to (12).

Consider now the case when $f = 1$. We have

$$\widetilde{V}_n^{\wedge}(1) - 2 = \frac{1}{\pi} \int_0^{2\pi} [T_h^{\theta} V_n^{\wedge}(1) - T_h^{\theta} 1] \, dh.$$

Consequently,

$$\left\| \widetilde{V}_n^{\wedge}(1) - 2 \right\| \leqslant 2 \left\| V_n^{\wedge}(1) - 1 \right\|. \tag{15}$$

By the hypothesis of the theorem, as $n \to \infty$ the right-hand side of inequality (15) tends to zero. Therefore from (15) follows the equality

$$\lim_{n \to \infty} \left\| \widetilde{V}_n^{\wedge}(1) - 2 \right\| = 0,$$

which is the particular case of equality (12) when $f = 1$.

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Note: Figure translations are in progress. See original paper for figures.