

ON INFINITELY DIVISIBLE LAWS WITH AN UNBOUNDED POISSON SPECTRUM

1963

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196301.77935>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

I. V. OSTROVSKII

ON INFINITELY DIVISIBLE LAWS WITH AN UNBOUNDED POISSON SPECTRUM

(Presented by Academician S. N. Bernstein, 24 V 1963)

Let F be an infinitely divisible law (i.d.l.), and let $\varphi(t)$ be its characteristic function (c.f.). Denote by Ω the class of i.d. laws for which $\ln \varphi(t)$ is represented in the form

$$\begin{aligned} \ln \varphi(t) = & i\beta t - \gamma t^2 + \sum_{m=-\infty}^{\infty} \lambda_m^{(1)} \left(e^{i\mu_m t} - 1 - \frac{i\mu_m t}{1 + \mu_m^2} \right) + \\ & + \sum_{m=-\infty}^{\infty} \lambda_m^{(2)} \left(e^{-i\nu_m t} - 1 + \frac{i\nu_m t}{1 + \nu_m^2} \right), \end{aligned}$$

where β is real, $\gamma \geq 0$; $\lambda_m^{(1)}, \lambda_m^{(2)} \geq 0$; $\mu_m, \nu_m > 0$ ($m = 0, \pm 1, \pm 2, \dots$), and the following conditions are fulfilled:

$$1. \quad \sum_{m=-\infty}^{\infty} \lambda_m^{(1)} \mu_m^2 (1 + \mu_m^2)^{-1} + \sum_{m=-\infty}^{\infty} \lambda_m^{(2)} \nu_m^2 (1 + \nu_m^2)^{-1} < \infty.$$

2. The numbers $\mu_{m+1}\mu_m^{-1}$ and $\nu_{m+1}\nu_m^{-1}$ are integers ($m = 0, \pm 1, \pm 2, \dots$). This class of laws was introduced and studied in depth in the works of Yu. V. Linnik, the results of which are set forth in Chs. VII-X of the book ⁽³⁾ (see also the bibliography there). Yu. V. Linnik proved ⁽³⁾, Ch. VIII that, if an i.d.l. has a Gaussian component, its belonging to Ω is a necessary condition for it to have only i.d. components. Whether the condition is sufficient has not yet been established. Yu. V. Linnik obtained the following result in this direction ⁽³⁾, Ch. X):

If an i.d.l. F belongs to the class Ω and there exist constants $c > 0$ and $a > 0$ such that, for all sufficiently large μ_m and ν_m ,

$$\ln \ln[(\lambda_m^{(1)})^{-1}] > c\mu_m^{1+a}, \quad \ln \ln[(\lambda_m^{(2)})^{-1}] > c\nu_m^{1+a}, \quad (1)$$

then F has only i.d. components.

Yu. V. Linnik conjectured (³, p. 257) that the restriction (1) can be replaced by the following weaker one:

$$\lambda_m^{(1)} = O(\exp\{-k\mu_m^2\}), \quad \lambda_m^{(2)} = O(\exp\{-k\nu_m^2\}) \quad \text{for every } k > 0. \quad (2)$$

Following Yu. V. Linnik, we shall call an entire function $\varphi(t)$ ridge-shaped if

$$|\varphi(\sigma + i\tau)| \leq |\varphi(i\tau)|, \quad -\infty < \sigma, \tau < \infty.$$

Obviously, every entire c.f. is ridge-shaped. The converse, generally speaking, is false (³, p. 63). Our main result is the following theorem.

Theorem 1. *If an i.d.l. F of the class Ω satisfies condition (2) and its c.f. $\varphi(t)$ admits the factorization*

$$\varphi(t) = \varphi_1(t)\varphi_2(t),$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are entire ridge-shaped functions, then $\varphi_1(t)$ and $\varphi_2(t)$ are, up to a constant factor, c.f.'s of i.d. laws.

If $F = F_1 * F_2$, then, denoting by $\varphi(t), \varphi_1(t), \varphi_2(t)$ the characteristic functions of F, F_1, F_2 , respectively, we shall have $\varphi(t) = \varphi_1(t)\varphi_2(t)$. If $F \in \Omega$ and condition (2) is fulfilled, then it is easy to see that $\varphi(t)$ is an entire function. By D. A. Raikov's theorem ([3], p. 58), the functions $\varphi_1(t)$ and $\varphi_2(t)$ are then also entire. Applying Theorem 1, we are readily convinced of the validity of Yu. V. Linnik's assertion.

We obtain Theorem 1 as a consequence of the following assertion.

Theorem 2. Let $f(t)$ be an entire function admitting the representation

$$f(t) = \sum_{p=1}^{\infty} \lambda_p^{(1)} e^{i\xi p t} + \sum_{p=1}^{\infty} \lambda_p^{(2)} e^{-i\eta p t} + l(it),$$

where $\xi > 0, \eta > 0$; $\lambda_p^{(1)}$ and $\lambda_p^{(2)}$ are real and satisfy, for any $k > 0$, the condition

$$|\lambda_p^{(1)}| + |\lambda_p^{(2)}| = O(\exp\{-kp^2\});$$

$l(z)$ is an entire function of exponential type, taking only real values for real z and satisfying, for some ξ', η' ($0 < \xi' < \xi, 0 < \eta' < \eta$), the conditions:

$$l(z) = O(\exp\{\xi' z\}), \quad \text{Im } z = 0, \quad \text{Re } z \rightarrow +\infty,$$

$$l(z) = O(\exp\{\eta'|z|\}), \quad \text{Im } z = 0, \text{Re } z \rightarrow -\infty,$$

$$l(z) = O(|z|^2), \quad \text{Re } z = 0, |z| \rightarrow \infty.$$

Suppose that the relation

$$\varphi_1(t)\varphi_2(t) = \exp\{f(t)\},$$

holds, where $\varphi_1(t)$ and $\varphi_2(t)$ are entire characteristic functions. Then ($r = 1, 2$)

$$\ln \varphi_r(t) = \sum_{p=1}^{\infty} P_{pr}^{(1)}(it)e^{i\xi pt} + \sum_{p=1}^{\infty} P_{pr}^{(2)}(it)e^{-imp t} + l_r(it) + id_r,$$

where: a) $P_{pr}^{(1)}(z)$ and $P_{pr}^{(2)}(z)$ are polynomials of degree not exceeding 4 with real coefficients; the coefficient largest in modulus $\lambda_{pr}^{(s)}$ of the polynomial $P_{pr}^{(s)}(z)$ satisfies, for any $k > 0$, the condition $\lambda_{pr}^{(s)} = O(\exp\{-kp^2\})$, $s = 1, 2$;
b) $l_r(z)$ are entire functions representable in the form

$$l_r(z) = z^5 \int_{-\eta'}^{\xi'} e^{tz} \psi_r(t) dt + P_r(z),$$

where $\psi_r(t)$ is absolutely continuous on $[-\eta', 0]$ and $[0, \xi']$, and $P_r(z)$ is a polynomial of degree not exceeding 4; c) d_r are real constants.

We outline the proof of Theorem 2.

Lemma 1. If $\varphi(t)$ is an entire characteristic function, then there is a constant $k_\varphi > 0$ such that

$$\ln |\varphi(i\tau)| > -k_\varphi(|\tau| + 1), \quad -\infty < \tau < \infty.$$

Lemma 2. If $\varphi(t)$ is an entire characteristic function without zeros, then $\varphi(t) = \exp\{\Phi(t) + id\}$, where $\Phi(t)$ is an entire function taking, for $\text{Re } t = 0$, only real values, and d is a real constant.

Denote by $\Phi_r(t)$, $r = 1, 2$, entire functions, real for $\text{Re } t = 0$, and such that $\varphi_r(t) = \exp\{\Phi_r(t) + id_r\}$, $r = 1, 2$ (d_r are real constants). We shall consider the function $g(z) = \Phi_1(-iz)$. For real x and y put $u(x, y) = \text{Re } g(x + iy)$; denote: $\varkappa_1(x) = u(x, 0) - u(x, 2\pi\xi^{-1})$, $\varkappa_2(x) = u(x, 0) - u(x, 2\pi\eta^{-1})$.

Lemma 3. For real x and y the following relations are valid:

$$\begin{aligned}
 0 \leq u(x, 0) - u(x, y) &\leq 2 \sum_{p=1}^{\infty} \lambda_p^{(1)} e^{\xi p x} \sin^2 \left(\frac{1}{2} \xi p y \right) + \\
 &+ 2 \sum_{p=1}^{\infty} \lambda_p^{(2)} e^{-\eta p x} \sin^2 \left(\frac{1}{2} \eta p y \right) + l(x) - \operatorname{Re} l(x + iy); \quad (3)
 \end{aligned}$$

$$\varkappa_1(x) = O(\exp\{\xi' x\}), \quad x \rightarrow +\infty; \quad \varkappa_2(x) = O(\exp\{\eta' |x|\}), \quad x \rightarrow -\infty. \quad (4)$$

Lemma 4. Whatever $k > 0$, for real x and y

$$u(x, y) = O((x^2 + y^2) \exp\{kx^2\}), \quad x^2 + y^2 \rightarrow \infty.$$

The estimate is obtained with the aid of (3) and Lemma 1.

Lemma 5. Whatever $k > 0$, the estimate holds

$$g(z) = O(|z|^3 \exp\{k(\operatorname{Re} z)^2\}), \quad |z| \rightarrow \infty.$$

Lemma 6. For the function $g(z)$ there is the representation

$$g(z) = z^5 \int_{-\infty}^{\infty} e^{zt} \Psi(t) dt + P(z), \quad (5)$$

where $\Psi(t)$ is a real function, absolutely continuous on $(-\infty, 0]$ and $[0, +\infty)$, satisfying, for every $k > 0$, the condition $\Psi(t) = O(\exp\{-kt^2\})$; $P(z)$ is a polynomial of degree not exceeding 4 with real coefficients.

Lemma 7. The function $u(x, y)$ is an entire function of x and y . For fixed $k > 0$ and y , the estimate holds in the complex x -plane

$$u(x, y) = O(|x|^3 \exp\{k(\operatorname{Re} x)^2\}), \quad |x| \rightarrow \infty.$$

The lemma follows from the relation

$$u(x, y) = \frac{1}{2} \{g(x + iy) + g(x - iy)\}$$

and Lemma 5.

Lemma 8. There are representations

$$\varkappa_1(x) = x^5 \int_{-\infty}^{\xi'} e^{xt} \Psi_1(t) dt + Q_1(x), \quad \varkappa_2(x) = x^5 \int_{-\eta'}^{\infty} e^{xt} \Psi_2(t) dt + Q_2(x), \quad (6)$$

where Ψ_1 is real and absolutely continuous on $(-\infty, 0]$ and $[0, \xi']$; Ψ_2 is real and absolutely continuous on $[-\eta', 0]$ and $[0, +\infty)$; for every $k > 0$ the estimate

$$|\Psi_1(-t)| + |\Psi_2(t)| = O(\exp\{-kt^2\}), \quad t \rightarrow +\infty;$$

holds; Q_1 and Q_2 are polynomials of degree not exceeding 4 with real coefficients.

To prove Lemma 8, from Lemma 7 and relations (4), with the aid of the Phragmén-Lindelöf principle, we conclude that the function $\varkappa_1(x)$ is of exponential type not exceeding ξ' in the half-plane $\operatorname{Re} x \geq 0$, and the function $\varkappa_2(x)$ is of exponential type not exceeding η' in the half-plane $\operatorname{Re} x \leq 0$. We then use the Paley-Wiener theorem.

Lemma 9. Let the functions $X_1(t)$ and $X_2(t)$ be continuous, and let $S_1(t)$ and $S_2(t)$ have q continuous derivatives for $-\infty < t < \infty$, and suppose that the conditions

1.
$$X_r(t) = O(e^{-2|t|}), \quad S_r^{(q)}(t) = O(e^{|t|}), \quad r = 1, 2.$$

2. On some interval $[a, b]$ ($-\infty < a < b < \infty$),

$$X_2(t) \equiv 0, \quad S_1(t) \neq 0.$$

- 3.

$$\int_{-\infty}^{\infty} (t^\rho S_1(t))^{(q)} X_1(t) dt = \int_{-\infty}^{\infty} (t^\rho S_2(t))^{(q)} X_2(t) dt, \quad \rho = 0, 1, 2, \dots$$

Then on the interval $[a, b]$ the function $X_1(t)$ is a polynomial of degree not exceeding $q - 1$. It is necessary to show that, for any function $h(t)$ that is q -times continuously differentiable on $[a, b]$ and satisfies the conditions $h^{(s)}(a) = h^{(s)}(b) = 0$ ($s = 0, 1, \dots, q$), the relation

$$\int_a^b h^{(q)}(t) X_1(t) dt = 0; \quad (7)$$

holds; after this the lemma follows easily from a known lemma of the calculus of variations ⁽¹⁾. To prove relation (7), one must consider a sequence of polynomials $\{Q_m(t)\}_{m=1}^{\infty}$ for which

$$\lim_{m \rightarrow \infty} \left\{ \sup_{-\infty < t < \infty} |H^{(q)}(t) - Q_m(t)| e^{-|t|} \right\} = 0,$$

where $H(t) = h(t)S_1^{-1}(t)$ for $t \in [a, b]$, and $H(t) = 0$ for $t \notin [a, b]$. Such a sequence of polynomials exists by S. N. Bernstein's theorem (2).

Theorem 2 is now obtained as follows. Taking the real part in (5), we arrive at the relations

$$\varkappa_1(x) = 2 \int_{-\infty}^{\infty} \frac{\partial^5}{\partial t^5} \left(e^{xt} \sin^2 \frac{\pi t}{\xi} \right) \Psi(t) dt + R_1(x),$$

$$\varkappa_2(x) = 2 \int_{-\infty}^{\infty} \frac{\partial^5}{\partial t^5} \left(e^{xt} \sin^2 \frac{\pi t}{\eta} \right) \Psi(t) dt + R_2(x),$$

where R_1 and R_2 are polynomials of degree not exceeding 4. Comparing these relations with relations (6), we obtain ($p = 0, 1, 2, \dots$)

$$\int_{-\infty}^{\infty} \left\{ t^p \left(2t^5 \sin^2 \frac{\pi t}{\xi} \right) \right\}^{(V)} \Psi(t) dt = \int_{-\infty}^{\xi'} \{t^p(t^5)\}^{(V)} \Psi_1(t) dt,$$

$$\int_{-\infty}^{\infty} \left\{ t^p \left(2t^5 \sin^2 \frac{\pi t}{\eta} \right) \right\}^{(V)} \Psi(t) dt = \int_{-\infty}^{\infty} \{t^p(t^5)\}^{(V)} \Psi_2(t) dt.$$

Hence, with the aid of Lemma 9, we conclude that $\Psi(t)$ is a polynomial of degree not exceeding 4 on each interval lying on the ray $\xi' < t < \infty$ and containing no integral multiples of ξ , and on each interval lying on the ray $-\infty < t < -\eta'$ and containing no integral multiples of η . Splitting the integral in (5) into integrals over the intervals:

$$\dots [-p\eta, -(p-1)\eta], \dots, [-2\eta, -\eta], [-\eta, -\eta'], [-\eta', \xi'], \\ [\xi', \xi], [\xi, 2\xi], \dots, [(p-1)\xi, p\xi], \dots,$$

we easily obtain what was to be proved.

I express my deep gratitude to Yu. I. Lyubich for substantial help in the proof of Lemma 9.

Kharkov State University
named after A. M. Gorky

Received
21 V 1963

References

- ¹ N. I. Akhiezer, *Lectures on the Calculus of Variations*, Moscow, 1955, p. 208.
- ² S. N. Bernstein, *Collected Works*, 1, Moscow, 1952, p. 278.
- ³ Yu. V. Linnik, *Decompositions of Probability Laws*, Leningrad, 1960.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.