HYDROMECHANICS

A. I. MOROZOV, L. S. SOLOV' EV

1963

SovietRxiv

Abstract

Full Text

HYDROMECHANICS

A. I. MOROZOV, L. S. SOLOV' EV

ON SYMMETRIC MAGNETOHYDRODYNAMIC FLOWS

(Presented by Academician M. A. Leontovich on 27 X 1962)

In the present paper a scheme is given for deriving the system of equations (10)–(12), describing stationary flows of an ideally conducting compressible fluid in a magnetic field under helical symmetry (^{1}), and some examples of plane and axisymmetric flows are considered.

1. The starting point is the system of equations of magnetic hydrodynamics for nondissipative processes (^{2}):

$$\rho(\mathbf{v}\nabla)\mathbf{v} = -\nabla p - [\mathbf{H}\ \mathrm{rot}\,\mathbf{H}] - \rho\nabla\Phi; \tag{1}$$

$$\mathbf{v}\nabla S = 0; \tag{2}$$

$$\mathrm{rot}[\mathbf{vH}] = 0; \tag{3}$$

$$\mathrm{div}\,\rho\mathbf{v} = 0; \tag{4}$$

$$\mathrm{div}\,\mathbf{H} = 0. \tag{5}$$

Here $\mathbf{v}$ is the velocity, ρ the density, p the pressure, S the entropy, $\mathbf{B} \equiv \sqrt{4\pi}\,\mathbf{H}$ the magnetic field, and Φ the potential of forces of non-electromagnetic origin.

From (2) follows the possibility of introducing the enthalpy W according to the equality $dW = dp/\rho$, valid along a streamline of the fluid. Under helical symmetry of the flow all quantities depend only on two variables: r and $\theta \equiv \varphi - \alpha z$, where r, φ, z are cylindrical coordinates, $L \equiv 2\pi/\alpha = \mathrm{const}$ is the pitch of the screw. The equality to zero of the divergence of the vectors $\rho\mathbf{v}$ and $\mathbf{H}$, and also $\mathbf{j}_0 \equiv \mathrm{rot}\,\mathbf{v}$ and $\mathbf{j} \equiv \mathrm{rot}\,\mathbf{H}$, makes it possible to introduce "current functions" ψ_0, ψ, I_0, I, defined by the equalities*

$$r\begin{pmatrix} v_r \\ H_r \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\rho}\dfrac{\partial\psi_0}{\partial\theta} \\ \dfrac{\partial\psi}{\partial\theta} \end{pmatrix}, \qquad \alpha r\begin{pmatrix} v_z \\ H_z \end{pmatrix} - \begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\rho}\dfrac{\partial\psi_0}{\partial r} \\ \dfrac{\partial\psi}{\partial r} \end{pmatrix},$$

$$r\begin{pmatrix} j_{0r} \\ j_r \end{pmatrix} = \begin{pmatrix} \dfrac{\partial I_0}{\partial\theta} \\ \dfrac{\partial I}{\partial\theta} \end{pmatrix}, \qquad \alpha r\begin{pmatrix} j_{0z} \\ j_z \end{pmatrix} - \begin{pmatrix} j_{0\varphi} \\ j_\varphi \end{pmatrix} = \begin{pmatrix} \dfrac{\partial I_0}{\partial r} \\ \dfrac{\partial I}{\partial r} \end{pmatrix}, \tag{6}$$

The functions introduced are characterized by the fact that the fluid streamlines lie on the surfaces $\psi_0(r,\theta) = \text{const}$, the magnetic lines of force on the "magnetic surfaces"$\psi(r,\theta) = \text{const}$, the electric-current lines on the surfaces $I(r,\theta) = \text{const}$, and the vortex lines on the surfaces $I_0(r,\theta) = \text{const}$, all these surfaces possessing helical symmetry. The function ψ is expressed through the components of the vector potential $\mathbf{A}$ by the formula $\psi = A_z + \alpha r A_\varphi$.

From equation (3) it follows that $\psi_0 = \psi_0(\psi)$, i.e. the fluid flows along magnetic surfaces. Introducing, for the symmetry, a new function ξ, we shall assume that $\psi_0 = \psi_0(\xi)$, $\psi = \psi(\xi)$.

* The resulting equations have a symmetric form with respect to $\mathbf{v}$ and $\mathbf{H}$, and it is convenient to write them in two-component form.

From the definitions of the vectors $\mathbf{j}_0$ and $\mathbf{j}$ we find

$$I_0 = v_z + \alpha r v_\varphi, \qquad I = H_z + \alpha r H_\varphi. \tag{7}$$

From (1) and (3), respectively, the Jacobian equalities are obtained

$$\frac{\partial(\psi, I)}{\partial(r,\theta)} = \frac{\partial(\psi_0, I_0)}{\partial(r,\theta)}, \qquad \frac{\partial(\psi, I_0/\beta)}{\partial(r,\theta)} = \frac{\partial(\psi_0, I/\beta\rho)}{\partial(r,\theta)}, \tag{8}$$

where $\beta \equiv 1 + \alpha^2 r^2$. According to (8) we obtain

$$I_0\psi_0' - I\psi' = A(\xi), \qquad I\psi_0'/\rho - I_0\psi' = \beta B(\xi). \tag{9}$$

Here a prime denotes differentiation with respect to ξ, while A and B are arbitrary quantities depending only on ξ, the first quantity being connected with conservation of momentum and the second with the frozen-in character of the magnetic field.

From equation (1), taking into account the relations written above, one derives an equation for ξ and an equation for W, which is an analogue of the Bernoulli integral in ordinary hydrodynamics.

The complete system of equations for the case of helical symmetry, with $S(\xi) =$ const, has the form

$$\frac{s}{\rho}\Delta^*\xi + \frac{1}{2\beta\rho}\frac{\partial s}{\partial\xi}(\nabla\xi)^2 - \frac{\psi_0'^2}{\beta\rho^3}(\nabla\rho\nabla\xi) + \frac{1}{2\beta\rho}\frac{\partial}{\partial\xi}\frac{A^2}{s} +$$

$$+\frac{\beta}{2}\frac{\partial}{\partial\xi}\frac{B^2}{s} + \frac{\partial}{\partial\xi}\frac{AB\psi_0'}{\rho s\psi'} - \frac{2\alpha A}{\beta^2\rho} - U' = 0; \tag{10}$$

$$W + \frac{v^2}{2} + \Phi + \frac{\beta B^2}{s} + \frac{AB\psi_0'}{\rho s\psi'} = U, \qquad s \equiv \frac{\psi_0'^2}{\rho} - \psi i'^2; \tag{11}$$

$$r\begin{pmatrix} v_r \\ H_r \end{pmatrix} = \begin{pmatrix} \psi_0'/\rho \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial\theta}, \qquad \alpha r\begin{pmatrix} v_z \\ H_z \end{pmatrix} - \begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = \begin{pmatrix} \psi_0'/\rho \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial r},$$

$$\begin{pmatrix} v_z \\ H_z \end{pmatrix} + \alpha r\begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = \frac{A}{s}\begin{pmatrix} \psi_0'/\rho \\ \psi' \end{pmatrix} + \frac{\beta B}{s}\begin{pmatrix} \psi' \\ \psi_0' \end{pmatrix}. \tag{12}$$

The "integrals of motion" ψ_0, ψ, A, B and U are arbitrary functions of ξ,

$$\Delta^* = \frac{1}{r}\frac{\partial}{\partial r}\frac{r}{\beta}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

and the partial derivative with respect to ξ is taken at fixed ρ.

As limiting cases of (10)–(12), one can obtain the corresponding equations for the plane problem ($\alpha \to 0$) and for the axially symmetric problem (as $\alpha \to \infty$).

In the case of translational symmetry, when there is no dependence on the coordinate z, we obtain

$$\frac{s}{\rho}\Delta\xi + \frac{1}{2\rho}\frac{\partial s}{\partial\xi}(\nabla\xi)^2 - \frac{\psi_0'^2}{\rho^3}(\nabla\rho\nabla\xi) +$$

$$+\frac{1}{2\rho}\frac{\partial}{\partial\xi}\frac{A^2}{s} + \frac{1}{2}\frac{\partial}{\partial\xi}\frac{B^2}{s} + \frac{\partial}{\partial\xi}\frac{AB\psi_0'}{\rho s\psi'} - U' = 0; \tag{13}$$

$$W + \frac{v^2}{2} + \Phi + \frac{B^2}{s} + \frac{AB\psi_0'}{\rho s\psi'} = U, \qquad s \equiv \frac{\psi_0'^2}{\rho} - \psi i'^2; \tag{14}$$

$$r\begin{pmatrix} v_r \\ H_r \end{pmatrix} = \begin{pmatrix} \psi_0'/\rho \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial\varphi}, \qquad \begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = -\begin{pmatrix} \psi_0'/\rho \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial r},$$

$$\begin{pmatrix} v_z \\ H_z \end{pmatrix} = \frac{A}{s}\begin{pmatrix} \psi_0'/\rho \\ \psi' \end{pmatrix} + \frac{B}{s}\begin{pmatrix} \psi' \\ \psi_0' \end{pmatrix}, \tag{15}$$

where Δ is the Laplace operator, and the function ψ is equal to A_z, the component of the vector potential. In order to represent (13)–(15) in Cartesian coordinates x, y, z, it is enough for the equations for the transverse components **v**

and **H** may be rewritten in the form

$$\begin{pmatrix} v_x \\ H_x \end{pmatrix} = \begin{pmatrix} \psi_0'/\rho \\ \psi' \end{pmatrix}\frac{\partial \xi}{\partial y}, \qquad \begin{pmatrix} v_y \\ H_y \end{pmatrix} = -\begin{pmatrix} \psi_0'/\rho \\ \psi' \end{pmatrix}\frac{\partial \xi}{\partial x}. \tag{15'}$$

In the case of **axial symmetry**, when there is no dependence on the azimuth φ, the system of equations of steady flow takes the form

$$\frac{s}{\rho}\Delta^*\xi+\frac{1}{2\rho r^2}\frac{\partial s}{\partial \xi}(\nabla\xi)^2-\frac{\psi_0'^2}{\rho^3 r^2}(\nabla\rho\nabla\xi)+\frac{1}{2\rho r^2}\frac{\partial}{\partial\xi}\frac{A^2}{s}+\frac{r^2}{2}\frac{\partial}{\partial\xi}\frac{B^2}{s}+\frac{\partial}{\partial\xi}\frac{AB\psi_0'}{\rho s\psi'}-U' = 0; \tag{16}$$

$$W + \frac{v^2}{2} + \Phi + \frac{r^2B^2}{s} + \frac{AB\psi_0'}{\rho s\psi'} = U, \qquad s \equiv \frac{\psi_0'^2}{\rho} - \psi i'^2; \tag{17}$$

$$r\begin{pmatrix} v_r \\ H_r \end{pmatrix} = -\begin{pmatrix} \psi_0'/\rho \\ \psi' \end{pmatrix}\frac{\partial \xi}{\partial z}, \qquad r\begin{pmatrix} v_z \\ H_z \end{pmatrix} = \begin{pmatrix} \psi_0'/\rho \\ \psi' \end{pmatrix}\frac{\partial \xi}{\partial r},$$

$$r\begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = \frac{A}{s}\begin{pmatrix} \psi_0'/\rho \\ \psi' \end{pmatrix} + \frac{r^2B}{s}\begin{pmatrix} \psi' \\ \psi_0' \end{pmatrix}. \tag{18}$$

Here, analogously to (10)–(15), ψ_0, ψ, A, B, U are functions depending only on ξ, which may be prescribed arbitrarily; the prime denotes differentiation with respect to ξ; the partial derivative with respect to ξ is taken at constant ρ, and

$$\Delta^* \equiv \frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial z^2}.$$

The function $\psi(r, z)$ is related to the azimuthal component of the vector potential by the relation $\psi = rA_\varphi$.

We note that in the special case $\psi' = 0$, the terms containing ψ' in the denominator in formulas (10)–(18) are absent.

Equations analogous to (13)–(18) for the case of an incompressible fluid ($\rho =$ const) were obtained in papers ([3],[4]), while the equations of helical flows analogous to (10)–(12) were obtained in paper ([5]).

2. Of substantial interest are problems on flows that vary slowly along one of the coordinates, which in a number of cases can be solved completely in the "adiabatic approximation" :

a) **For plane flow**, independent of the coordinates x and z, the system (13) –(15) has the integral $p + H^2/2 = \text{const}^*$. In the adiabatic approximation, discarding terms of order $(\partial\xi/\partial x)^2$ and $\partial^2\xi/\partial x^2$, and eliminating U from equations (13) and (14), we obtain

$$\frac{\partial}{\partial y}\left(p + \frac{H^2}{2}\right) = 0.$$

It follows that

$$p + \frac{H^2}{2} = \frac{f^2(x)}{2}, \tag{19}$$

where $f(x)$ is a slowly varying function of x.

For example, in the case $\psi' = 0,\ A = 0,\ W = 0\ (\xi = \psi_0)$, equations (19) and (14) allow the problem to be reduced to the quadrature

$$\int_0^\xi \frac{B\,d\xi}{\sqrt{U - fB}} = \sqrt{2}\,f\{y - Y(x)\}. \tag{20}$$

Here U and B are arbitrary functions of ξ, and $Y(x)$ is a slow function of x.

b) **For axisymmetric flow**, analogously to the preceding case, in the adiabatic approximation of slow variation along z (neglecting

* In what follows we set $\Phi = 0$.

neglecting terms of order $(\partial\xi/\partial z)^2$ and $\partial^2\xi/\partial z^2$, we obtain

$$\frac{\partial}{\partial r}\left(p + \frac{H^2}{2}\right) + \frac{1}{r}\left(H_\varphi^2 - \rho v_\varphi^2\right) = 0. \tag{21}$$

This equation is not integrable in general form; however, in some particular cases it can also be used to obtain first integrals of the system (16)–(18).

Let us consider the flow when $\psi' = 0$, $A = 0$ $(\xi = \psi_0)$; then, according to (21), we have

$$\frac{\partial W}{\partial r} + B\frac{\partial}{\partial r}\left(\rho r^2 B\right) = 0. \tag{22}$$

The last equation is integrated in the cases $W = 0$ and $B = \text{const}$. In the first case we obtain $\rho r^2 B = f(z)$, and, using (17), arrive at the quadrature

$$\int_0^{\xi} \frac{B\,d\xi}{\sqrt{U - fB}} = \sqrt{2}\,f \ln \frac{r}{R(z)}, \tag{23}$$

where $R(z)$ is an arbitrary (slow) function of z. In the second case the first integral is $W + \rho r^2 B^2 = f(z)$. Determining from this $\rho = \rho(r, z)$ and substituting in (17), we also arrive at a quadrature in which, however, ρ is an implicit function of r.

c) In a number of cases the adiabatic integral of the system (16)–(18) can be obtained even without relation (21). For example, when ψ_0', ψ', A, B, U are constants, equation (16) has the integral $\frac{s}{r}\frac{\partial \xi}{\partial r} = f(z)$. Setting $W = 0$ and determining ρ from equation (17), we find

$$\psi {i'}^2 \rho = \psi_0'^2 \pm \frac{\psi_0'}{r} \sqrt{\frac{(A\psi' + r^2 B\psi_0')^2 + r^2 f^2 \psi {i'}^2}{r^2 B^2 + 2U\psi {i'}^2}}. \tag{24}$$

Hence for ξ we obtain the quadrature

$$\xi = \frac{f}{\psi {i'}^2} \left\{ \mp\psi_0' \int_R^r \sqrt{\frac{r^2 B^2 + 2U\psi {i'}^2}{(A\psi' + r^2 B\psi_0')^2 + r^2 f^2 \psi {i'}^2}}\, r^2 dr - \frac{r^2 - R^2}{2} \right\}, \tag{25}$$

expressible in terms of elliptic integrals.

We note that the solutions thus obtained make it possible simply to determine the shape of the channel, whose boundaries are the surfaces $\xi = \text{const}$.

Received
11 X 1962

REFERENCES

1. L. S. Solov' ev, Dissertation, Moscow Engineering-Physics Institute, 1962.
2. L. D. Landau, E. M. Lifshitz, *Electrodynamics of Continuous Media*, 1957.
3. S. Chandrasekhar, *Astrophys. J.*, **124**, 232 (1956).
4. V. S. Tkalich, *Izv. AN SSSR, OTN, Mechanics and Mechanical Engineering*, No. 4 (1959).

5. L. S. Solov’ ev, *ZhTF*, **31**, 4 (1961).

Note: Figure translations are in progress. See original paper for figures.