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Abstract

Full Text

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Ω-Semigroups, Ω-Rings, and Representations

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1. Just as, in the classical theory of representations, representations of groups are connected with representations of rings, so representations of automorphism groups of arbitrary (universal) algebras are naturally connected with the consideration of Ω-semigroups and their representations. The present note is devoted to questions related to this.

Let G be some Ω-algebra—an algebra with underlying set G and a system of fundamental operations Ω, and let $S = S(G)$ be the set of all transformations (single-valued mappings into itself) of the set G. Along with the ordinary multiplication of transformations, which is not connected with the system Ω, all operations from Ω are also transferred to S. Let, for example, ω be an n-ary operation from Ω, and let $\varphi_1, \varphi_2, \dots, \varphi_n$ belong to S. Then $\varphi_1\varphi_2\cdots\varphi_n\omega$ is the transformation defined by the formula $g(\varphi_1\varphi_2\cdots\varphi_n)\omega = (g\varphi_1)(g\varphi_2)\cdots(g\varphi_n)\omega$ for any $g \in G$. Thus S becomes an Ω-algebra and, moreover, S also has multiplication. It is easy to see that this multiplication has the property of left distributivity with respect to all operations of the system Ω: if ω is an n-ary operation from Ω, $\varphi, \varphi_1, \varphi_2, \dots, \varphi_n \in S$, then

$$\varphi \cdot (\varphi_1\varphi_2\cdots\varphi_n)\omega = (\varphi\varphi_1)(\varphi\varphi_2)\cdots(\varphi\varphi_n)\omega.$$

Right distributivity does not hold in general, but it holds for those φ that are endomorphisms of the algebra G.

We now introduce the following definition. A set Σ is called an **Ω-semigroup** if multiplication is defined on this set, with respect to which Σ is a semigroup (the multiplicative semigroup Σ), and also some set of algebraic operations Ω is defined (the Ω-algebra Σ), with left distributivity of multiplication holding with respect to all operations of the system Ω. The distributive elements of such an Ω-semigroup are the elements for which not only the left but also the right distributive rule holds. Obviously, the set of all distributive elements of an Ω-semigroup Σ is a subsemigroup in the multiplicative semigroup Σ. We shall denote this set by $D(\Sigma)$.

It is easy to see that if Φ is some subsemigroup in $D(\Sigma)$, then the Ω-subalgebra of the Ω-algebra Σ generated by the elements of Φ is closed also with respect to multiplication—it is an Ω-subsemigroup. In particular, an Ω-subsemigroup in S is the subalgebra generated by all endomorphisms of the algebra G. The elements of such a subalgebra will be called **quasi-endomorphisms** of the algebra G.

If an Ω-semigroup Σ is generated by its distributive elements, then we shall say that such an Ω-semigroup has a distributive basis. If $\Sigma = D(\Sigma)$, then such an Ω-semigroup is called distributive. We note that a number of results concerning distributive Ω-semigroups were obtained by Ya. V. Khion. We shall now give one condition under which distributivity of an Ω-semigroup follows from the existence of a distributive basis.

Let ω_1 and ω_2 be two operations from Ω, ω_1 n-ary and ω_2 m-ary, and let the elements x_{ij} of Σ form a matrix with n rows and m columns. The operations ω_1 and ω_2 are called **permutable** (on Σ) if, for any such matrix of elements from Σ, applying ω_2 to the rows followed by applying ω_1 to the results leads to the same element of Σ as applying ω_1 to the columns followed by applying to ⋯

to all elements ω_2 obtained. Thus, for example, in a commutative Ω-group every operation of the system Ω commutes with addition (^{1}).

An algebra G with a system of operations Ω is called **commutative** if any two operations from Ω, including identical ones, commute. It is not difficult to verify that if the algebra G is commutative, then the set of all endomorphisms of this algebra is an Ω-subsemigroup in the Ω-semigroup S, i.e. in this case every quasiendomorphism turns out to be an endomorphism.

If Σ is an Ω-semigroup with a distributive basis and the Ω-algebra Σ is commutative, then Σ is a distributive Ω-semigroup.

2. Let now G be an Ω-algebra and let Σ be some Ω-semigroup. A **representation** of Σ with respect to G is any homomorphism $\mu\Sigma$ into the Ω-semigroup $S(G)$. Specifying a representation of Σ with respect to G is equivalent to specifying an action of Σ in G, which we denote by ∘, such that the following conditions are satisfied:

$$1. \quad (g \circ \sigma) \circ \varphi = g \circ \sigma\varphi,$$

$$2. \quad g \circ (\varphi_1 \dots \varphi_n \omega) = (g \circ \varphi_1)(g \circ \varphi_2) \dots (g \circ \varphi_n)\omega$$

for all $g \in G$, $\sigma, \varphi \in \Sigma$, $\omega \in \Omega$, and corresponding tuples of arguments φ_i.

Such a pair (G, Σ), together with the given representation, will be called an **Ω-pair**. In the usual way one defines homomorphisms, isomorphisms and equivalence of Ω-pairs, as well as subpairs. An example of a representation is the regular representation of an Ω-semigroup. If Σ is an Ω-semigroup and φ and σ are its elements, then the formula $\varphi \circ \sigma = \varphi\sigma$ defines the regular representation of the Ω-semigroup Σ with respect to the Ω-algebra Σ.

If a pair (G, Σ) is given, then an element $\sigma \in \Sigma$ is called **strictly distributive** if $\sigma \in D(\Sigma)$ and σ acts in G as an endomorphism. The totality of all strictly distributive elements of Σ forms a subsemigroup in $D(\Sigma)$. We denote it by

$D^*(\Sigma)$. The representation is called strict if $D^*(\Sigma)$ generates all of Σ. It is clear that in the regular representation $D^*(\Sigma)$ coincides with $D(\Sigma)$.

Each representation $\mu\Sigma$ with respect to G determines a congruence π of the Ω-semigroup Σ: for any σ and φ from Σ, $\sigma\pi\varphi$ if and only if for every $g \in G$ one has $g \circ \sigma = g \circ \varphi$. Such a congruence π is called the **congruence of the given representation**. As usual, a representation is called **faithful** if the congruence of the representation is trivial. Let us also note that if the π-congruence of the Ω-algebra Σ is invariant with respect to right multiplication by elements of Σ, then the regular representation of Σ induces a factor-representation of Σ with respect to the factor-algebra Σ/π.

In what follows, let (G, Σ) be some Ω-pair. For each $g \in G$ denote by g^Σ the set of all elements of G of the form $g \circ \sigma$, $\sigma \in \Sigma$.

It is easy to see that g^Σ is always a Σ-admissible subalgebra in G, and that the mapping $\sigma \mapsto g \circ \sigma$ is a homomorphism of the Ω-algebra Σ onto the subalgebra g^Σ. Let $\pi(g)$ be the congruence corresponding to this homomorphism. This congruence is invariant with respect to right multiplications and, consequently, one may speak of the factor-representation of Σ with respect to the Ω-algebra $\Sigma/\pi(g)$. Further assigning to each $g \circ \sigma$ from g^Σ the coset containing the element σ, we shall show that the representation of Σ with respect to g^Σ is equivalent to the representation of Σ with respect to $\Sigma/\pi(g)$. Let us emphasize also that every distributive element of Σ acts in g^Σ as an endomorphism.

If the representation of Σ with respect to G is faithful, then, obviously, the intersection of all $\pi(g)$ over all $g \in G$ is the trivial congruence of the Ω-algebra Σ, so that in this case the algebra Σ is isomorphic to a subdirect product of subalgebras of G, and every identity relation that holds in G also holds in Σ.

3. Let us define the free semigroup (group) Ω-semigroup. Let Γ be some semigroup and let $\mathfrak{A}$ be the free Ω-algebra with generating set Γ. We define in $\mathfrak{A}$ also a multiplication. If there are nullary operations in Ω, then to these operations there correspond special symbols $0_\alpha, 0_\beta, \ldots$. For each such 0_α and each $a \in \mathfrak{A}$ we put $a \cdot 0_\alpha = 0_\alpha$, and for each $u \in \Gamma$: $0_\alpha u = 0_\alpha$. Further, let $u \in \Gamma$, $u_1, u_2, \ldots, u_n$ be elements of Γ or symbols of nullary operations, and let ω be an n-ary operation from Ω, $n \geq 1$. Put

$$u_1 u_2 \ldots u_n \omega \cdot u = (u_1 u)(u_2 u) \ldots (u_{nu})\omega.$$

If now $a = b_1 b_2 \ldots b_n \omega$ and all $b_i u$ have already been defined, then, by definition,

$$a \cdot u = (b_1 u)(b_2 u) \ldots (b_{nu})\omega.$$

Thus the product $a \cdot u$ will be defined for any $a \in \mathfrak{A}$ and $u \in \Gamma$. Analogously we define

$$a \cdot (u_1 u_2 \ldots u_n)\omega = (au_1)(au_2) \ldots (au_n)\omega,$$

and if $b = (c_1 c_2 \ldots c_n)\omega$ and all $a \cdot c_i$ are defined, then

$$a \cdot b = (ac_1)(ac_2) \ldots (ac_n)\omega.$$

> We obtain a rule for multiplying arbitrary elements of $\mathfrak{A}$. In this case, by the definition of the operation, left distributivity holds and all elements of Γ are distributive elements. It is also easy to verify the associativity of multiplication. Thus $\mathfrak{A}$ becomes an Ω-semigroup, moreover with a distributive basis. This is the free semigroup Ω-semigroup. If Γ is a group, then we have a group Ω-semigroup.

Now suppose that a representation of the group Γ by automorphisms of an Ω-algebra G is given, and let $\mathfrak{A}$ be the corresponding group Ω-semigroup. Putting, for any $g \in G$ and any 0_α,

$$g \circ 0_\alpha = e_\alpha,$$

where e_α is the corresponding distinguished element in G, in the usual way we can extend the representation of Γ to a representation of $\mathfrak{A}$. The elements of Γ will then be strictly distributive elements, and the representation itself will be strict.

The possibility of extending a representation of Γ to a representation of $\mathfrak{A}$, in particular, means that all cyclic representations of the group Γ with respect to Ω-algebras can be realized on the material of the Ω-semigroup $\mathfrak{A}$. Recall that a representation of Γ with respect to G is called cyclic if G is generated by elements of the form $g \circ \sigma$ with fixed $g \in G$ and all possible $\sigma \in \Gamma$. Suppose further that we have in mind the study of representations with respect to algebras belonging to some primitive class of Ω-algebras K. In that case one should start from the reduced free group Ω-semigroups described above.

Let $\mathfrak{A}$ be the free group Ω-semigroup of the group Γ, and let θ be the congruence in the Ω-algebra $\mathfrak{A}$ corresponding to the identities defining the class K. It is not hard to see that θ is also a congruence of the Ω-semigroup $\mathfrak{A}$. Thus one can speak of the reduced free group Ω-semigroup $\mathfrak{A}/\theta$. In this case, obviously, one may regard Γ as contained in $\mathfrak{A}/\theta$. It is easy to understand that if a representation of $\mathfrak{A}$ with respect to an algebra G belonging to the class K is given, then the congruence θ belongs to the congruence of the representation π. Hence it follows that the action of $\mathfrak{A}$ in G induces an action of $\mathfrak{A}/\theta$ in G, which is an extension of the representation of the group Γ with respect to G.

4. The consideration of representations of groups with respect to Ω-groups leads to the concept of an Ω-ring. Let Ω_+ and $\Omega_\times$ be systems of operations consisting of the operations of the system Ω and, in addition, of addition and, respectively, multiplication. Let all operations from Ω, and also addition and multiplication, be defined on the set U. The set U, with respect to these operations, is called an Ω-ring if U is an $\Omega_\times$-group and an Ω_+-semigroup. If G is an Ω-group, then $S(G)$ naturally becomes an Ω-ring. In the case of the empty set Ω, an Ω-ring is almost a ring.

Let $(G, \mathfrak{U})$ be an Ω_+-pair, consisting of an Ω-group G and an Ω-ring $\mathfrak{U}$. Now, instead of the congruences π, $\pi(g)$, and θ considered above, one may start from the corresponding kernels of homomorphisms. The kernel corresponding to the congruence π is the kernel of the representation. It is an ideal in the $\Omega_\times$-group $\mathfrak{U}$. The kernels corresponding to the congruences $\pi(g)$ are right ideals.

Now let A and B, $A \subset B$, be two $\mathfrak{U}$-admissible Ω-subgroups in G.

We shall say that A is an admissible ideal in B if A is an ideal in B and, for any $a \in A$, $b \in B$, and $u \in \mathfrak{U}$, the condition $(a+b)\circ u - b\circ u \in A$ is satisfied. If A is an admissible ideal in B, then the original representation $\mathfrak{U}$ with respect to G induces a representation of $\mathfrak{U}$ with respect to the quotient group B/A. Let us note that in the case when the Ω_+-semigroup $\mathfrak{U}$ has a strictly distributive basis, the fact that A is an ideal in B already implies that A is an admissible ideal. We shall call a pair A, B a composition pair if A is an admissible ideal in B and the representation of $\mathfrak{U}$ with respect to B/A is an irreducible representation. In this connection, a representation is called irreducible if in the corresponding Ω-group there are no nontrivial admissible normal systems (we shall assume that the zero subgroup is always admissible).

A representation of $\mathfrak{U}$ with respect to G is called regular if, for any two admissible Ω-subgroups A and B of G, the fact that A is a maximal admissible subgroup in B implies that A is an admissible ideal in B. It is clear that if the representation of $\mathfrak{U}$ with respect to G is regular, then for any composition pair A, B in the quotient group B/A there are no nontrivial admissible Ω-subgroups.

We shall further call the radical of the representation $\mathfrak{U}$ with respect to G the intersection of all kernels of representations of $\mathfrak{U}$ with respect to B/A over all composition pairs A, B in G. We denote this radical by $\alpha(\mathfrak{U}, G)$. The radical of an Ω-ring $\mathfrak{U}$ is called the radical of the regular representation of this Ω-ring. It is easy to see that in the case when $\mathfrak{U}$ is a ring, the radical thus defined coincides with the Jacobson radical. In the case of a regular representation, for the radical $\alpha(\mathfrak{U}, G)$ a characterization analogous to the characterization of the Jacobson radical is possible.

We shall assume that $\mathfrak{U}$ has a multiplicative identity ε, which acts in G as the identity transformation. An element $u \in \mathfrak{U}$ will be called an externally quasiregular element if, for every $g \in G$, there exists $v \in \mathfrak{U}$ such that $g\circ(\varepsilon - u)v = g$.

The following assertion is true: if the representation of $\mathfrak{U}$ with respect to G is regular, then every element of the radical $\alpha(\mathfrak{U}, G)$ is an externally quasiregular element. If, moreover, the Ω_+-semigroup $\mathfrak{U}$ has a strictly distributive basis, then every right ideal of $\mathfrak{U}$ consisting of externally quasiregular elements belongs to $\alpha(\mathfrak{U}, G)$.

Just as for Ω-rings, one defines the radical of a representation of a group with respect to an Ω-group and regular representations of groups; moreover, if an Ω-ring is generated by a group, then the corresponding connections arise here.

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REFERENCES

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Note: Figure translations are in progress. See original paper for figures.