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Abstract

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DECOMPOSITION INTO IRREDUCIBLE REPRESENTATIONS OF THE RESTRICTION OF REPRESENTATIONS OF THE PRINCIPAL SERIES OF THE PROPER LORENTZ GROUP TO THE REAL LORENTZ GROUP

(Presented by Academician L. S. Pontryagin on 14 III 1963)

I. The real Lorentz group G_0 is a subgroup of the proper Lorentz group G_+; G_0 consists of all those transformations of 4-dimensional space (3 Cartesian coordinates and a time coordinate) that do not change one of the Cartesian coordinates. In the present work the question of decomposition is solved only for the principal series of irreducible representations of G_+.

The group G_+ is locally isomorphic to the group G of all complex matrices of the second order with determinant equal to one. G_+ is a homomorphic image of G ([1]), §9. Under this homomorphism, in G_0 there passes the subgroup of all real matrices in G. In representation theory, instead of G_+ one considers G, and G_0 is replaced by the corresponding group of real matrices, which we shall also denote by G_0.

The author uses the notation of the papers ([1,2]).

II. Let us explain the formulation of the problem. We consider the principal series of unitary representations of the group G ([1]), §10 in the Hilbert space $L_2(z)$ of functions $f(z)$, defined on the complex plane z, measurable, with summable square modulus with respect to the measure $dz = dx\,dy$, where $x = \operatorname{Re} z$, and $y = \operatorname{Im} z$.

Let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$; T_g denotes the operator of the representation under consideration corresponding to g. According to ([1]), §10,

$$T_g f(z) = (\beta z + \delta)^{-m} \, |\beta z + \delta|^{m+i\rho-2} \, f(\widetilde{zg}), \tag{1}$$

where $f(z) \in L_2(z)$, and

$$\widetilde{zg} = \frac{\alpha z + \gamma}{\beta z + \delta}, \tag{2}$$

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$m = 0, \pm 1, \pm 2, ...$, and ρ is any real number. m and ρ define a representation from the principal series.

The totality of the operators T_g for $g \in G_0$ defines a certain representation of the group G_0, which is called the **restriction of the initial representation to** G_0. We denote this representation of the group G_0 by $\mathfrak{S}_{m,\rho}$. The problem consists in decomposing $\mathfrak{S}_{m,\rho}$ into irreducible representations.

III. The solution of this problem follows from the fact that the representation $\mathfrak{S}_{m,\rho}$ in the case $m \neq 0$ is equivalent to a certain part of the regular representation of the group G_0, i.e. $L_2(z)$ can be mapped isometrically onto a certain subspace P of the space $L_2(G_0)$ (P depends on m) in such a way that the representation $\mathfrak{S}_{m,\rho}$ goes over into the representation by right translations on the group G_0. The representation $\mathfrak{S}_{0,\rho}$ is equivalent to the tensor product of two representations, each of which is a certain part of the regular representation.

It is clear that the given method of solution essentially uses the Plancherel formula for the group G_0. We shall use the results of the paper ([2]), where this formula is actually obtained. In addition, we shall be interested in the irreducible representations of the principal series of the group G_0, since the decomposition of the representation $\mathfrak{S}_{m,\rho}$ is carried out with respect to them. Therefore, before formulating the final result, we shall describe these representations.

IV. In the paper ([2]) a realization of the group G_0 is used in the form of the group G_1 of complex matrices $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$, where $\bar{\alpha}$ is the number complex-conjugate to α, $\alpha\bar{\alpha} - \beta\bar{\beta} = 1$. The connection between the groups G_0 and G_1 is established by the formula

$$a_0 = t^{-1} g_0 t, \qquad \text{where } t = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \qquad g_0 \in G_0, \quad a_0 \in G_1. \tag{3}$$

The group G_1 has two continuous and two discrete series of irreducible representations, which belong to the class of principal series. The continuous series are denoted respectively by C_q^0 and $C_q^{1/2}$; each representation of these series is uniquely determined by a number $q \geq 1/4$.

The discrete series are denoted by D_k^+ and D_k^-; a representation from these series is determined by a number $k = 1,\ ^3/2,\ 2, ...$

A representation of the series C_q^0 and $C_q^{1/2}$ is realized in the Hilbert space $\mathfrak{H}$ of all functions $f(\Phi)$, measurable on the interval $[-\pi, \pi]$ with square-integrable modulus with respect to the measure $d\Phi/2\pi$. The scalar product of vectors g and $f \in \mathfrak{H}$ has the form

$$(g, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{g(\Phi)}\, f(\Phi)\, d\Phi. \tag{4}$$

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The operator of the representation C_q^0 ($C_q^{1/2}$) is denoted by $T_{is}(a)$ ($T'_{is}(a)$), where $s = \sqrt{q - 1/4}$, and has the form

$$T_{is}(a)f(\Phi) = |\bar{\alpha} - \beta e^{i\Phi}|^{-1-2is} f(a^{-1}\Phi),$$

$$T'_{is}(a)f(\Phi) = (\bar{\alpha} - \beta e^{i\Phi})^{-1}|\bar{\alpha} - \beta e^{i\Phi}|^{-2is} f(a^{-1}\Phi), \tag{5}$$

where $a = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$, and the transformation $\Phi \mapsto a\Phi$ on the interval $[-\pi, \pi]$ is determined from the condition

$$e^{i(a\Phi)} = \frac{\bar{\alpha} e^{i\Phi} + \bar{\beta}}{\alpha + \beta e^{i\Phi}} \quad ((^2),\ \S 6,\ 7).$$

Representations of the series D_p^+ and D_p^- are realized in the Hilbert space $\mathfrak{H}_{2p}$ of functions $f(w)$, analytic in the unit disk of the complex plane ([2], §9). The scalar product of functions f and $g \in \mathfrak{H}_{2p}$ is given by the formula

$$(f, g) = \frac{2p-1}{\pi} \int_{|w|<1} (1 - w\bar{w})^{2p-2} \overline{f(w)}\, g(w)\, dw, \tag{6}$$

where $dw = du\, dv$ for $u = \operatorname{Im} w$ and $v = \operatorname{Im} w$.

The operator $T_p^+(a)$ of the representation D_p^+ has the form

$$T_p^+(a)f(w) = (-\beta w + \bar{\alpha})^{-2p} f\left(\frac{\alpha w - \bar{\beta}}{-\beta w + \bar{\alpha}}\right). \tag{7}$$

The operator $T_p^-(a)$ of the representation D_p^- is defined by the formula

$$T_p^-(a) = T_p^+(\bar{a}) \tag{7a}$$

for $\bar{a} = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \beta & \alpha \end{pmatrix}$, if $a = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$.

V. The representation $\mathfrak{S}_{m,\rho}$ is uniquely determined by the number m and does not depend on ρ, i.e. $\mathfrak{S}_{m,\rho_1}$ and $\mathfrak{S}_{m,\rho_2}$ are equivalent for $\rho_1 \neq \rho_2$. We indicate formulas that effect the decomposition of the representation $\mathfrak{S}_{m,\rho}$ into irreducible representations. Put

$$b = \frac{1}{2\sqrt{\operatorname{Im} z}} \begin{pmatrix} i\bar{z} + 1 & \bar{z} + i \\ z - i & -iz + 1 \end{pmatrix}, \qquad \operatorname{Im} z > 0. \tag{8}$$

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It is easy to see that $b \in G_1$. Denote

$$K_m(z,\Phi,s) = \begin{cases} \dfrac{1}{2\sqrt{\pi}}\,|\operatorname{Im} z|^{\frac{i\rho}{2}-1}\,T_{is}(b^{-1})e^{i\frac{m}{2}\Phi}, & \text{for even } m,\\ \dfrac{1}{2\sqrt{\pi}}\,|\operatorname{Im} z|^{\frac{i\rho}{2}-1}\,T'_{is}(b^{-1})e^{i\frac{m-1}{2}\Phi}, & \text{for odd } m, \end{cases} \tag{9}$$

where $m = 0, \pm 1, \ldots$, and $T_{is}(a)$ and $T'_{is}(a)$ are defined by (5);

$$K^{\pm}_{mp}(z,w) = (-1)^{m-p}\sqrt{\frac{\pi(m+2p-1)!}{(2p-1)(m-p)!(p-1)!}}\,|\operatorname{Im} z|^{\frac{i\rho}{2}-1}T^{\pm}_p(b^{-1})w^{m-p}, \tag{10}$$

where $m = p, p+1, \ldots;\quad p = 1,\ {}^3/_2,\ 2, \ldots;$

$$\Lambda = \begin{cases} +, & \text{if } m>0,\\ -, & \text{if } m<0, \end{cases} \qquad m = \pm 2, \pm 3, \ldots;$$

$$p = \begin{cases} \left|\dfrac{m}{2}\right|,\ \left|\dfrac{m}{2}\right| - 1, \ldots, 1, & \text{if } m \text{ is even},\\ \left|\dfrac{m}{2}\right|,\ \left|\dfrac{m}{2}\right| - 1, \ldots,\ {}^3/_2, & \text{if } m \text{ is odd}. \end{cases} \tag{11}$$

Theorem. Let $f(z) \in L_2(z)$. Then the integrals

$$g_p(w) = \int_{\operatorname{Im} z>0} \overline{K^{\Lambda}_{|\frac{m}{2}|p}(z,w)}\, f(z)\,dz, \qquad \tilde{g}_p(w) = \int_{\operatorname{Im} z<0} \overline{K^{\Lambda}_{|\frac{m}{2}|p}(z,w)}\, f(z)\,dz \tag{12}$$

converge in $\mathfrak{H}_{2p}$ for every p in (11). If $\mathfrak{H}$ ($\mathfrak{H}'$) denotes the Hilbert space of all measurable functions $f(\Phi; s)$ for $\Phi \in [-\pi, \pi]$, $s \in [0, +\infty)$ with summable square of the modulus with respect to the measure $d\Phi\,ds$ with weight $\dfrac{\operatorname{cth}\pi s}{s}$ $\left(\dfrac{\operatorname{th}\pi s}{s}\right)$, then the integrals

$$f(\Phi,s) = \frac{1}{2\sqrt{\pi}}\int_{\operatorname{Im} z>0} K_m(z,\Phi,s) f(z)\,dz, \tag{13}$$

$$\bar{f}(\Phi,s) = \frac{1}{2\sqrt{\pi}}\int_{\operatorname{Im} z<0} K_m(\bar{z},\Phi,s) f(z)\,dz \tag{14}$$

converge in $\mathscr{H}'$ for odd m and converge in $\mathscr{H}$ for even m. Moreover, for almost all z,

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$$f(z) = \begin{cases} \sum_p' \frac{2p-1}{\pi} \int_{|w|<1} K^{\Lambda}_{|\frac{m}{2}|p}(z,w) g_p(w)(1-\overline{w}w)^{2p-2}\,dw \\ \quad + 2\sqrt{\pi} \int_0^{+\infty} \int_{-\pi}^{\pi} K_m(z,\Phi,s) f(\Phi,s)\,d\Phi\,ds, & \text{for } \operatorname{Im} z > 0; \\ \sum_p' \frac{2p-1}{\pi} \int_{|w|<1} K^{\Lambda}_{|\frac{m}{2}|p}(\overline{z},w) \tilde{g}_p(w)(1-\overline{w}w)^{2p-2}\,dw \\ \quad + 2\sqrt{\pi} \int_0^{+\infty} \int_{-\pi}^{\pi} K_m(\overline{z},\Phi,s) \tilde{f}(\Phi,s)\,d\Phi\,ds, & \text{for } \operatorname{Im} z < 0, \end{cases} \tag{15}$$

where $\sum_p'$ denotes summation over all p in (11):

$$\int |f(z)|^2\,dz = \sum_p' \left[(g_p(w), g_p(w)) + (\tilde{g}_p(w), \tilde{g}_p(w))\right] + (f(\Phi,s), f(\Phi,s)) + (\tilde{f}(\Phi,s), \tilde{f}(\Phi,s)). \tag{16}$$

When $f(z)$ passes into $T_{g_0} f(z)$, $g_p(w)$ and $\tilde{g}_p(w)$ are transformed respectively into $T_p^{\Lambda}(a_0)g$ and $T_p^{-\Lambda}(a_0)\tilde{g}$, while $f(\Phi,s)$ and $\tilde{f}(\Phi,s)$—respectively into $T_{is}(a_0)f$, $T_{is}(a_0)\tilde{f}$ or $T'_{is}(a_0)f$, $T'_{is}(a_0)\tilde{f}$.

Here f and $\tilde{f} \in \mathcal{H}$ $(\mathcal{H}')$ for almost every fixed $s \in [0,+\infty)$, if the function f $(\tilde{f})$ is regarded as a function of Φ. Thus, in the space $\mathcal{H}$ $(\mathcal{H}')$, a direct continuous sum of the representations C_q^0 $(C_q^{1/2})$ is realized. Formula (16) is the Plancherel formula for the representation $\mathfrak{S}_{m,\rho}$. This formula may be rewritten in the form

$$L_2(z) = \mathfrak{h}_{|m|} \oplus \mathfrak{h}_{|m|-2} \oplus \cdots \oplus \mathfrak{h}_2 \oplus \mathcal{H} \oplus \mathfrak{h}_{|m|} \oplus \mathfrak{h}_{|m|-2} \oplus \cdots \oplus \mathfrak{h}_2 \oplus \mathcal{H} \quad \text{for even } m; \tag{17}$$

$$L_2(z) = \mathfrak{h}_{|m|} \oplus \mathfrak{h}_{|m|-2} \oplus \cdots \oplus \mathfrak{h}_3 \oplus \mathcal{H}' \oplus \mathfrak{h}_{|m|} \oplus \mathfrak{h}_{|m|-2} \oplus \cdots \oplus \mathfrak{h}_3 \oplus \mathcal{H}' \quad \text{for odd } m.$$

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2. V. Bargmann, Ann. Math., **48**, 568 (1947).

Note: Figure translations are in progress. See original paper for figures.

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