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Abstract

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THEORY OF THE STATIONARY PROPERTIES OF A FULLY IONIZED NEAR-EARTH PLASMA

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Are the radiation belts around the Earth a structure of a single statistical formation? To answer this question it is important to take account of the following: 1) the ellipsoidality of the distribution functions at great altitudes; 2) the reaction of this distribution to the action of the external (dipole) magnetic field of the Earth; 3) the interaction of charged particles with one another and with the total charge of the Earth and atmosphere. The requirement that the distribution functions be ellipsoidal is dictated by the rejection of rigid-body rotation of the atmosphere at great altitudes, with simultaneous preservation of the temperature distribution. The indicated factors lead to strong spatial anisotropy in the distribution of electrons and nuclei, which can be calculated in advance and compared with the experimentally observed data on the Earth' s radiation belts.

The main characteristics of the particle distribution are determined by the ellipsoidality constant A. For its numerical determination it is sufficient to know the concentrations and temperatures of the particles at only one spatial point. The quantitative formulation of the problem and the method of solution are entirely analogous to ([1]).

The distribution functions of nuclei and electrons are described by the system of equations:

$$\frac{\partial f_\pm}{\partial t} + v_r \frac{\partial f_\pm}{\partial r} + \frac{v_\vartheta}{r}\frac{\partial f_\pm}{\partial \vartheta} + \frac{v_\varphi}{r\sin\vartheta}\frac{\partial f_\pm}{\partial \varphi} + \left(\frac{v_\vartheta^2 + v_\varphi^2}{r}\right)\frac{\partial f_\pm}{\partial v_r} +$$

$$+\left(-\frac{v_r v_\vartheta}{r} + \operatorname{ctg}\vartheta\,\frac{v_\varphi^2}{r}\right)\frac{\partial f_\pm}{\partial v_\vartheta} + \left(-\frac{v_r v_\varphi}{r} - \operatorname{ctg}\vartheta\,\frac{v_\vartheta v_\varphi}{r}\right)\frac{\partial f_\pm}{\partial v_\varphi} -$$

$$-\frac{e}{m_\pm}\left\{\left[-\frac{\partial V}{\partial r} + \frac{1}{c}\left(v_\vartheta H_\varphi - v_\varphi H_\vartheta\right)\right]\frac{\partial f_\pm}{\partial v_r} +\right.$$

$$+\left[-\frac{\partial V}{r\,\partial\vartheta} + \frac{1}{c}\left(\left(v_\varphi H_r - v_r H_\varphi\right)\right)\right]\frac{\partial f_\pm}{\partial v_\vartheta} +$$

$$+\left[-\frac{\partial V}{r\sin\vartheta\,\partial\varphi}+\frac{1}{c}\left(v_r H_\vartheta - v_\vartheta H_r\right)\right]\frac{\partial f_\pm}{\partial v_\varphi}\right\} = 0,$$

$$\Delta V = -4\pi e\int (f_+ - f_-)\,d\mathbf{v}, \qquad \operatorname{rot}\mathbf{H} = \frac{4\pi e}{c}\int \mathbf{v}(f_+ - f_-)\,d\mathbf{v},$$

where **H** includes the self-field of the currents and the external dipole magnetic field of the Earth.

The boundary conditions of the problem are:

$$V\big|_{r\to\infty}\to 0, \qquad H_{r,\vartheta,\varphi}\big|_{r\to\infty}\to 0, \qquad V\big|_{r\to a}\to \mathcal{E}/a, \qquad \iint (f_+ - f_-)\,d\mathbf{r}\,d\mathbf{v} < \infty.$$

Here a is the radius and $\mathcal{E}$ is the electric charge of the Earth. The integral condition requires the finiteness of the charge falling on the entire near-Earth fully ionized plasma.

We shall seek a stationary temperature distribution:

$$f = \rho(r,\vartheta)\,w(v_r^2)\,w(v_\vartheta^2)\,w(v_\varphi^2)\,F(r\sin\vartheta,\ v_\varphi),$$

where the functions ρ, w, and F are to be determined.

Restricting ourselves to the symmetry of the problem, i.e., assuming $\partial/\partial\varphi = 0$ and $H_\varphi = 0$, we find

$$\rho = \rho_0\exp(-eV/\theta), \qquad w = (m/2\pi\theta)^{1/2}\exp(-mv^2/2\theta),$$

$$r\,\partial F/\partial r = (v_\varphi - reH_\vartheta/mc)\,\partial F/\partial v_\varphi, \qquad \partial F/\partial\vartheta = (\operatorname{ctg}\vartheta\,v_\varphi + reH_r/mc)\,\partial F/\partial v_\varphi.$$

The solution of the equations for F is expressed in terms of an arbitrary function depending in a special way on the arguments:

$$F = F\left[r\sin\vartheta\,(v_\varphi - (e/mc)A_\varphi)\right],$$

where A_φ is the projection of the vector potential, if the conditions $rA_\varphi|_{r\to\infty}\to 0$, $A_\varphi\sin\vartheta|_{\vartheta\to 0}$ are used.

In order that the solution be at the same time a temperature solution and contain the rotation of the system—not only rigid-body rotation, but also its generalization—it is necessary to specify the function F as follows:

$$F = e^{Q}, \qquad Q = (m\omega/2\theta)\, 2r' v'_\varphi A r {-}^2 v'^2_\varphi, \qquad r' = r\sin\vartheta, \qquad v'_\varphi = v_\varphi - \frac{e}{mc} A_\varphi,$$

where ω, θ, and A are constants whose meaning is determined below.

Represent the product $w(v'^2_\varphi)F$ in the form of a function with a shift in velocity space in the direction of the parallels and with a "potential function" S:

$$w(v'^2_\varphi)F(r\sin\vartheta\, v'_\varphi) = (m/2\pi\theta)^{1/2} \exp\{-(m/2\theta + \alpha)(v_\varphi - u_\varphi)^2 + S\};$$

we find:

$$\alpha = A(r\sin\vartheta)^2, \qquad u_\varphi = \frac{(m/2\theta)\omega r\sin\vartheta + A(r\sin\vartheta)^2(e/mc)A_\varphi}{m/2\theta + A(r\sin\vartheta)^2},$$

$$S = \frac{(m/2\theta)^2}{m/2\theta + A(r\sin\vartheta)^2}\left\{\omega^2 r^2\sin^2\vartheta - 2\omega r\sin\vartheta\left(\frac{e}{mc}A_\varphi\right) - \frac{2\theta}{m}A(r\sin\vartheta)^2\left((e/mc)A_\varphi\right)^2\right\}.$$

We find the concentration distribution for both kinds of particles:

$$\rho_\pm(r,\vartheta) = \int f_\pm\, dv = \rho_0\left(1 + (2\theta/m)_\pm A_\pm r^2\sin^2\vartheta\right)^{-1/2}\exp\{-eV/\theta_\pm + S_\pm\}.$$

It is convenient for what follows to introduce dimensionless quantities:

$$1/x = eM\omega/c\theta_-\, r, \qquad \varphi = -eV/\theta_-, \qquad \psi = -(e\omega/c\theta_-)A_\varphi r, \qquad \mathcal{E}^* = \mathcal{E}/(M\omega/c).$$

Here M is the magnetic moment of the Earth, with respect to which it is convenient to express the quantities; $\mathcal{E}$ is the charge of the Earth. We also introduce the following dimensionless coefficients:

$$\lambda = (4\pi e^2\rho_0/\theta_-)(eM\omega/c\theta_-)^2, \qquad \varepsilon = (eM\omega^2/c^2\theta_-)^2, \qquad \alpha_\pm = (m/2\theta)_\pm\omega^2(eM\omega/c\theta_-)^2,$$
$$\gamma_\pm = [(2\theta/m)A]_\pm(eM\omega/c\theta_-)^2, \qquad \beta_\pm = (A\theta/m^2)_\pm(\theta_-/\omega^2).$$

Knowledge of the stationary distribution function makes it possible to formulate the problem of the distribution of electrons and nuclei by means of the potentials φ and ψ:

$$\Delta\varphi = \lambda\left[\frac{1}{\sqrt{\eta_+}}\exp\frac{\theta_-}{\theta_+}(\varphi + \Phi^+) - \frac{1}{\sqrt{\eta_-}}\exp(-\varphi + \Phi^-)\right],$$

$$\Phi^{\pm} = \pm\frac{\sin\vartheta}{\eta}\psi - \beta\frac{\sin^2\vartheta}{\eta}\psi^2 + \frac{\theta}{\theta_-}\alpha x^2\frac{\sin^2\vartheta}{\eta},$$

$$\mathrm{rot}_\varphi\,\mathrm{rot}\left(\mathbf{i}_\varphi\frac{\psi}{x}\right) = \lambda\varepsilon\left[\frac{1}{\sqrt{\eta_+}}u_\varphi^+\exp\frac{\theta_-}{\theta_+}(\varphi+\Phi^+) - \frac{1}{\sqrt{\eta_-}}u_\varphi^-\exp(-\varphi+\Phi^-)\right],$$

where $\eta = 1+\gamma x^2\sin^2\vartheta$, and α,β,γ, and θ each take two values, $+$ and $-$.

The unknown functions are subject to the conditions

$$\varphi|_{x\to\infty}\to 0,\qquad \psi|_{x\to\infty}\to 0,\qquad \varphi|_{x\to\infty}\to \mathcal{E}/a,$$

$$\mathcal{E}_s = \frac{\lambda}{2}\int_0^\pi\int_0^\infty\left[\frac{1}{\sqrt{\eta_+}}\exp(\varphi+\Phi^+)\frac{\theta_-}{\theta_+} - \frac{1}{\sqrt{\eta_-}}\exp(-\varphi+\Phi^-)\right]x^2\,dx\,\sin\vartheta\,d\vartheta < \infty.$$

Convergence of the integral that appears requires certain conditions on the parameters of the system.

Since φ and ψ tend to zero as $x\to\infty$, the integrand tends to the value $(1/\sqrt{\gamma_+})\exp(\alpha/\gamma)_+ - (1/\sqrt{\gamma_-})\exp(\alpha/\gamma)_-$. In view of the fact that the ellipsoidal constant enters into γ, but does not enter into α, one should assume that α and γ are independent; then from the convergence condition it follows that $\alpha_+ = \alpha_-$ and $\gamma_+ = \gamma_-$, or $(\theta/m)_+ = (\theta/m)_-$, $A_+ = A_-$. Thus, the strong non-isothermality of the near-Earth plasma is a consequence of a natural boundary condition.

The problem posed admits the following simplifications: 1) for all distances the strong inequality $\alpha x^2/(1+\gamma x^2)\ll 1$ holds, which makes it possible to neglect the potential function of the centrifugal forces; 2) the magnetic interaction of the currents is a small quantity, since the quantity characterizing this interaction is small: $\varepsilon = (eM\omega^2/c^2\theta_-)^2 \ll 1$. Therefore we shall seek the solution in the form of a series $\psi = \sin\vartheta/x + \varepsilon\psi_1 + \ldots$, where the first term represents the magnetic dipole field of the Earth. In the zeroth approximation with respect to ε we have

$$\Delta\varphi =$$

$$= \lambda\frac{1}{\sqrt{\eta}}\left[\exp\frac{\theta_-}{\theta_+}(\varphi+\Phi^+) - \exp(-\varphi+\Phi^-)\right],$$

$$\Phi^{\pm} = \pm\frac{\sin^2\vartheta}{x\eta} - \beta_{\pm}\frac{\sin^4\vartheta}{x^2\eta}.$$

Figure 1. Lines of maxima for the concentration of nuclei and electrons (case $\lambda < 1,\ 0 < \mathcal{E}^* < 1$)

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Since ρ_0, which enters into the parameter λ—the particle density as $x \to \infty$ (in interplanetary space)—depends on the activity of the Sun, there arises an eigenvalue problem for the nonlinear equation for the potential. Let us consider two limiting cases.

Case $\lambda < 1$. We shall seek the solution in the form of a series $\varphi = \varphi_0 + \lambda\varphi_1 + \lambda^2\varphi_2 + \ldots$, where φ_0 is the potential produced by charges outside the region under consideration, in our case by the total electric charge of the Earth and atmosphere.

It is convenient to introduce three distance zones $1 < x < 1/\sqrt{\lambda} < 1/\sqrt{\gamma} < \infty$. In the first zone, in the zeroth and first approximations, we have

$$\Delta\varphi_0 = 0, \qquad \Delta\varphi_1 = \left[\exp\frac{\theta_-}{\theta_+}(\varphi_0 + \Phi_0^+) - \exp(-\varphi_0 + \Phi_0^-)\right], \ldots,$$

$$\varphi_0 = \frac{\mathcal{E}^*}{x}, \qquad \Phi_0^\pm = \pm\frac{\sin^2\vartheta}{x} - \beta_\pm\frac{\sin^4\vartheta}{x^2}.$$

In the first approximation the concentrations of charged particles are expressed as follows:

$$\rho_\pm^{(1)} = \rho_0\exp(\theta_-/\theta_+)\left(\mp\mathcal{E}^*/x \pm \sin^2\vartheta/x - \beta_\pm\sin^4\vartheta/x^2\right).$$

These functions have a maximum, whose position is determined by the formulas:

$$x_+ = 2\beta_+\sin^4\vartheta/(-\mathcal{E}^* + \sin^2\vartheta), \qquad x_- = 2\beta_-\sin^4\vartheta/(\mathcal{E}^* - \sin^2\vartheta).$$

If the total charge of the Earth and atmosphere is positive but relatively small ($\mathcal{E}^* < 1$), then maxima exist in the concentration of nuclei and electrons, as functions of distance, of a definite form (see Fig. 1).

In the second and third distance zones, screening of the electrostatic potential arises, with different screening laws.

Case $\lambda > 1$. We use the expansion $\varphi = \varphi_0 + \varphi_1/\lambda + \varphi_2/\lambda^2 + \cdots$; from the original equation for φ we find, for the zero approximation,

Fig. 2. Lines of equal concentrations of the quantity $(\rho_+^1 - \rho_-^1)/\rho_0$ (case $\lambda > 1$

Figure 2: Fig. 2. Lines of equal concentrations of the quantity $(\rho_+^1 - \rho_-^1)/\rho_0$ (case $\lambda > 1$

$$\varphi_0 = -\frac{\sin^2\vartheta}{x\eta} - \frac{\beta_- - (\theta_-/\theta_+)\beta_+}{1+\theta_-/\theta_+}\,\frac{\sin^4\vartheta}{x^2\eta},$$

$$\eta = 1 + \gamma x^2 \sin^2\vartheta$$

and for the first approximation

$$\varphi_1 = \sqrt{\eta}\left(1+\frac{\theta_-}{\theta_+}\right)^{-1}\Delta\varphi_0 \exp\left(\frac{(\theta_-/\theta_+)(\beta_- + \beta_+)\sin^4\vartheta}{(1+\theta_-/\theta_+)x^2\eta}\right).$$

Determining from this the concentrations of charged particles, we conclude that in the zero approximation

$$\rho_+^{(0)} = \rho_0 \exp\left\{-\frac{(\theta_-/\theta_+)(\beta_- + \beta_+)}{1+\theta_-/\theta_+}\,\frac{\sin^4\vartheta}{x^2\eta}\right\} = \rho_-^{(0)}$$

they are equal and, with increasing x, tend to a constant value. In the following first approximation we find for the relative concentration

$$(\rho_+^{(1)} - \rho_-^{(1)})/\rho_0 = \Delta\varphi_0/\lambda,$$

where φ_0 is the zero approximation. The right-hand side as a function of ϑ and x is analyzed graphically. Identifying the boundary of the termination of positive charges ($\Delta\varphi_0 = 0$) with the boundary of the first radiation belt on the side of large distances, one can determine the value of the ellipsoidal constant A. From the conditions $\Delta\varphi_0 = 0$, $\vartheta = \pi/2$, assuming that for the indicated boundary, in order of magnitude, $r \sim 1 \cdot 10^9$ cm, $\theta_- \sim 10^3$ eV, we find $A \sim 1.3 \cdot 10^{-40}$ sec^2/cm^4.

From the graphical analysis it follows:

1. There is a belt with a predominance of nuclei over electrons (see Fig. 2). The quantity $(\rho_+^{(1)} - \rho_-^{(1)})/\rho_0$ reaches values of order 10^6 in it. The angle ϑ from which it begins is equal to $70^\circ 30'$.

Fig. 2. Lines of equal concentrations of the quantity $(\rho_+^1 - \rho_-^1)/\rho_0$ (case $\lambda > 1$)

2. Outside this belt, electrons are distributed mainly, since here $\Delta\varphi_0 < 0$. Approximately in the same angular range ϑ there is a definite maximum for electrons immediately beyond the boundary of the first belt.

The solution obtained satisfies the condition of neutrality of the entire system as a whole (the Earth, the atmosphere, and the fully ionized near-Earth plasma).

In all the preceding consideration it was assumed that the axes of rotation and of the magnetic moment coincide. If $\delta\vartheta$ denotes the angle between these directions, then the function ψ, which determines the vector potential of the Earth' s dipole, is expressed through the angle $\delta\vartheta$: $\psi = \sin(\vartheta + \delta\vartheta)/x$. Therefore, in all calculations it is necessary to make the replacement $\sin^2\vartheta \to \sin\vartheta \sin(\vartheta + \delta\vartheta)$.

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References

1. A. A. Vlasov, ZhTF, No. 7, 795 (1961).

Note: Figure translations are in progress. See original paper for figures.