

# CRITERIA FOR REGULAR FUNCTIONS TO BELONG TO TWO CLASSES OF FUNCTIONS OF TWO COMPLEX VARIABLES

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**Abstract**

**Full Text**

**MATHEMATICS**

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## **CRITERIA FOR REGULAR FUNCTIONS TO BELONG TO TWO CLASSES OF FUNCTIONS OF TWO COMPLEX VARIABLES**

*(Presented by Academician M. A. Lavrent'ev, 23 III 1963)*

In the classical theory of functions of one complex variable, an important role is played by two closely related classes of functions regular in the disk  $|z| < 1$ ,  $f(z) = z + c_2 z^2 + \dots$ , which map  $|z| < 1$  univalently, respectively, onto a domain starlike with respect to the origin and onto a convex domain. Effective necessary and sufficient conditions are well known <sup>(1)</sup> for functions  $f(z) = z + c_2 z^2 + \dots$ , regular in  $|z| < 1$ , to belong to these classes.\*

In the present note, in the case of two complex variables, definitions are given of two classes of functions  $M_D$  and  $N_D$ , whose nature is similar to the nature of the corresponding classes of functions of one complex variable indicated above, and effective necessary and sufficient conditions are established for functions  $F(w, z)$ , regular in the domain  $D$ ,  $F(0, 0) = 1$ , **to belong to the classes  $M_D$  and  $N_D$ . From these conditions it follows directly that the classes of functions considered in the author's papers <sup>(2,3)</sup> coincide respectively with the classes  $M_{D_1}$ \* and  $N_{D_1}$ .** Therefore all estimates obtained in the classes of functions studied in <sup>(2,3)</sup> are, respectively, estimates in the classes  $M_{D_1}$  and  $N_{D_1}$ .

Let the function  $F(w, z)$ ,  $F(0, 0) = 1$ , be regular in the domain  $D$ . Take the set  $D \cap \{w = k_0 z\}$  ( $k_0$  is a fixed finite number from the set of all complex numbers), i.e., the set which is the intersection of the domain  $D$  with the plane  $w = k_0 z$ .

**Definition 1.** We shall say that in the section  $D \cap \{w = k_0 z\}$  the function  $zF(w, z)$  is **starlike univalent** (or **convex univalent**) if the function  $zF(k_0 z, z)$ , as a function of one complex variable  $z$ , maps the corresponding disk\*\*\*\* univalently onto a domain starlike with respect to the origin (or onto a convex domain).

Considering the section of the domain  $D$  by the plane  $z = k'_0 w$  ( $k'_0$  is a fixed finite number from the set of all complex numbers), we analogously introduce

**Definition 1'.** We shall say that in the section  $D \cap \{z = k'_0 w\}$  the function  $wF(w, z)$  is **starlike univalent** (or **convex univalent**) if the function  $wF(w, k'_0 w)$ , as a function of one complex variable  $w$ , maps the corresponding

disk\*\*\*\*\* univalently onto a domain starlike with respect to the origin (or onto a convex domain).

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\* Of course, all this also holds in the case of the disk  $|z| < R$ ,  $R < \infty$ . In what follows in this note, when referring to the indicated results from the theory of functions of one complex variable, we shall have in mind precisely the case of the disk  $|z| < R$ .

\*\*  $D$  is a bounded complete bicircular domain (containing its center  $(0, 0)$ ).

\*\*\*  $D_1$  is that domain  $D$  whose boundary is twice continuously differentiable and analytically convex outward, while the curve corresponding in the "absolute quadrant-plane" to the boundary of  $D_1$  is convex or straight.

\*\*\*\* Here, by the corresponding disk is meant the projection of the section  $D \cap \{w = k_0 z\}$  onto the plane  $w = 0$ ; if  $k_0 = 0$ , then this will be the disk which the domain  $D$  cuts out of the plane  $w = 0$ .

\*\*\*\*\* Here, by the corresponding disk is meant the projection of the section  $D \cap \{z = k'_0 w\}$  onto the plane  $z = 0$ ; if  $k'_0 = 0$ , then this will be the disk which the domain  $D$  cuts out of the plane  $z = 0$ .

**Definition 2.** Denote by  $M_D (N_D)$  the class of functions  $F(w, z)$ , regular in the domain  $D$ ,  $F(0, 0) = 1$ , having the following properties: 1) in the section of the domain  $D$  by each plane from the set of analytic planes  $w = kz^*$  the function  $zF(w, z)$  is starlike univalent (convex univalent); 2) in the section  $D \cap \{z = 0\}$  the function  $wF(w, 0)$  is starlike univalent (convex univalent).

Let in the domain  $D$  the function  $F(w, z)$ ,  $F(0, 0) = 1$ , be regular. We shall show that then from properties 1) and 2) of Definition 2 it follows that: a) in the section of the domain  $D$  by each plane from the set of analytic planes  $z = k'w^{**}$  the function  $wF(w, z)$  is starlike univalent (convex univalent); b) in the section  $D \cap \{w = 0\}$  the function  $zF(0, z)$  is starlike univalent (convex univalent), and conversely.

Indeed, since for all  $k' \neq 0$

$$\operatorname{Re} \left( \frac{w (wF(w, k'w))'_w}{wF(w, k'w)} \right) = \operatorname{Re} \left( \frac{z (zF(z/k', z))'_z}{zF(z/k', z)} \right),$$

$$\operatorname{Re} \left( \frac{w (wF(w, k'w))''_{ww}}{(wF(w, k'w))'_w} \right) + 1 = \operatorname{Re} \left( \frac{z (zF(z/k', z))''_{zz}}{(zF(z/k', z))'_z} \right) + 1,$$

it follows from 1), taking Definitions 1 and 1' into account, that a) holds for all  $k' \neq 0$  (1). Further, from 2) we obtain a) for  $k' = 0$ , and from 1) for  $k = 0$  we obtain b). The converse assertion is established in the same way. Thus, properties 1), 2) and a), b) are equivalent.

**Theorem.** In order that a function  $F(w, z)$ , regular in the domain  $D$ ,  $F(0, 0) = 1$ , belong to the class  $M_D$ , respectively  $N_D$ , it is necessary and sufficient that in  $D$

$$\operatorname{Re} \left( \frac{L[F(w, z)]}{F(w, z)} \right) > 0, \quad (1)$$

respectively

$$\operatorname{Re} \left( \frac{L[L[F(w, z)]]}{L[F(w, z)]} \right) > 0, \quad (2)$$

where

$$L[F(w, z)] \equiv F(w, z) + wF'_w(w, z) + zF'_z(w, z).$$

**Proof.** Suppose that  $F(w, z)$  is a function of the class  $M_D$  ( $N_D$ ). Let  $(w_0, z_0)$  be any point of the domain  $D$ , distinct from the points  $(w, 0)$  of this domain, and let  $w = k_0z$  be the plane containing the point  $(w_0, z_0)$ . Such a plane exists, since there exists  $k_0$  for which  $w_0 = k_0z_0$ . In fact, taking  $k_0 = w_0/z_0$ , we obtain the required plane  $w = k_0z$ . By property 1) of the class  $M_D$  ( $N_D$ ), the function  $zF(w, z)$  in the section  $D \cap \{w = k_0z\}$  is starlike univalent (convex univalent), and hence, by Definition 1, the function regular in the corresponding disk

$$zF(k_0z, z) = z + \dots$$

maps this disk univalently onto a domain starlike with respect to the origin (onto a convex domain). Therefore, in this disk,

$$\operatorname{Re} \left( \frac{z(zF(k_0z, z))'_z}{zF(k_0z, z)} \right) > 0 \quad \left( \operatorname{Re} \left( \frac{z(zF(k_0z, z))''_{zz}}{(zF(k_0z, z))'_z} \right) + 1 > 0 \right)$$

\*  $k$  ranges over the entire set of complex numbers, except  $\infty$ .

\*\*  $k'$  ranges over the entire set of complex numbers, except  $\infty$ .

\*\*\* The regularity of the function  $zF(k_0z, z)$  in the corresponding disk and the indicated normalization of this function follow in an obvious way from the fact that, by the hypothesis of the theorem, the function  $F(w, z)$  is regular in the domain  $D$  and normalized by the condition  $F(0, 0) = 1$ . We shall keep this in mind in the subsequent course of the proof of the theorem.

or

$$\operatorname{Re} \left( \frac{L[\dot{F}(k_0 z, z)]}{F(k_0 z, z)} \right) > 0 \quad \left( \operatorname{Re} \left( \frac{L[L[F(k_0 z, z)]]}{L[F(k_0 z, z)]} \right) > 0 \right),$$

whence, for  $z = z_0$ , we have

$$\operatorname{Re} \left( \frac{L[F(w_0, z_0)]}{F(w_0, z_0)} \right) > 0 \quad \left( \operatorname{Re} \left( \frac{L[L[F(w_0, z_0)]]}{L[F(w_0, z_0)]} \right) > 0 \right).$$

The point  $(w_0, z_0)$  is an arbitrary point of the domain  $D$ , distinct from the points  $(w, 0)$  of this domain. Hence condition (1) (condition (2)) holds for all points of  $D$ , except for the points  $(w, 0)$  of the domain  $D$ . By property 2) of the class  $M_D(N_D)$ , the function  $wF(w, 0)$  in the section  $D \cap \{z = 0\}$  is starlike univalent (convex univalent), and therefore, by definition 1', the function regular in the disk  $D \cap \{z = 0\}$

$$wF(w, 0) = w + \dots^*$$

maps this disk univalently onto a domain starlike with respect to the origin (onto a convex domain). Therefore <sup>(1)</sup>, for the points  $(w, 0)$  of the domain  $D$ ,

$$\operatorname{Re} \left( \frac{w(wF(w, 0))'_w}{wF(w, 0)} \right) > 0 \quad \left( \operatorname{Re} \left( \frac{w(wF(w, 0))''_{ww}}{(wF(w, 0))'_w} + 1 \right) > 0 \right)$$

or

$$\operatorname{Re} \left( \frac{L[F(w, 0)]}{F(w, 0)} \right) > 0 \quad \left( \operatorname{Re} \left( \frac{L[L[F(w, 0)]]}{L[F(w, 0)]} \right) > 0 \right).$$

Thus, condition (1) (condition (2)) holds for all points of the domain  $D$ .

Conversely, suppose that condition (1) (condition (2)) is satisfied in the domain  $D$ . Let  $w = k_0 z$  be any plane among all possible analytic planes  $w = kz^{**}$ . Since condition (1) (condition (2)) holds in the domain  $D$ , it also holds in the section  $D \cap \{w = k_0 z\}$ , and therefore in the corresponding disk we have:

$$\operatorname{Re} \left( \frac{L[F(k_0 z, z)]}{F(k_0 z, z)} \right) > 0 \quad \left( \operatorname{Re} \left( \frac{L[L[F(k_0 z, z)]]}{L[F(k_0 z, z)]} \right) > 0 \right)$$

or, what is the same,

$$\operatorname{Re} \left( \frac{z(zF(k_0 z, z))'_z}{zF(k_0 z, z)} \right) > 0$$

$$\left( \operatorname{Re} \left( \frac{z[z(zF(k_0z, z))'_z]'_z}{z(zF(k_0z, z))'_z} \right) > 0 \right) \tag{3}$$

or

$$\operatorname{Re} \left( \frac{z(zF(k_0z, z))''_{zz}}{(zF(k_0z, z))'_z} (+1 > 0^{***}) \right).$$

Consequently <sup>(1)</sup>, the function regular in the corresponding disk,

$$zF(k_0z, z) = z + \dots$$

maps this disk univalently onto a domain starlike with respect to the origin (onto a convex domain), i.e., according to definition 1, in the section  $D \cap \{w = k_0z\}$  the function  $zF(w, z)$  is starlike univalent (convex univalent). But the plane  $w = k_0z$  is an arbitrary plane—

\* The regularity of the function  $wF(w, 0)$  in the disk  $D \cap \{z = 0\}$  and the indicated normalization of this function follow in an obvious way from the fact that, by the hypothesis of the theorem, the function  $F(w, z)$  is regular in the domain  $D$  and normalized by the condition  $F(0, 0) = 1$ . We shall keep this in mind also in the further course of the proof of the theorem.

\*\*  $k$  runs through the entire set of complex numbers, except  $\infty$ .

\*\*\*  $(zF(k_0z, z))'_z \neq 0$  in the corresponding disk, since condition (3) means that the function regular in the corresponding disk,

$$z(zF(k_0z, z))'_z = z + \dots$$

maps this disk univalently onto a domain starlike with respect to the origin (1), and therefore in this disk it has the unique simple zero  $z = 0$ .

...ness from all possible analytic planes  $w = kz$ . Thus, property 1) of the class  $M_D(N_D)$  is established.

Next, from condition (1) (condition (2)) in the disk  $D \cap \{z = 0\}$  we have

$$\operatorname{Re} \left( \frac{L[F(w, 0)]}{F(w, 0)} \right) > 0 \quad \left( \operatorname{Re} \left( \frac{L[L[F(w, 0)]]}{L[F(w, 0)]} \right) > 0 \right)$$

or, equivalently,

$$\operatorname{Re} \left( \frac{w(wF(w, 0))'_w}{wF(w, 0)} \right) > 0,$$

$$\left( \operatorname{Re} \left( \frac{w[w(wF(w, 0))'_w]'_w}{w(wF(w, 0))'_w} \right) > 0 \right) \quad (4)$$

or

$$\operatorname{Re} \left( \frac{w(wF(w, 0))''_{w^2}}{(wF(w, 0))'_w} + 1 > 0^* \right).$$

Consequently <sup>(1)</sup>, the function  $wF(w, 0) = w + \dots$ , regular in the disk  $D \cap \{z = 0\}$ , maps this disk univalently onto a domain star-shaped with respect to the origin (onto a convex domain), i.e., according to definition 1', in the section  $D \cap \{z = 0\}$  the function  $wF(w, 0)$  is starlike univalent (convex univalent). Thus property 2) of the class  $M_D$  ( $N_D$ ) is also established. The theorem is proved.

**Remark 1.** In exactly the same way as the classes  $M_D$  and  $N_D$ , one defines the classes  $M_E$  and  $N_E$ , where  $E$  is the bicylinder  $\{|w| < R_1, |z| < R_2\}$ . In the classes  $M_E$  and  $N_E$  the theorem established above holds. The proof is exactly the same as in the case of the domain  $D$ .

**Remark 2.** In the papers <sup>(2,3)</sup> I considered two classes of functions  $F(w, z)$ ,  $F(0, 0) = 1$ , regular in the domain  $D_1$  (or in the bicylinder  $E$ ), satisfying in  $D_1$  (or  $E$ ) respectively conditions (1), (2). From the theorem proved (Remark 1) it follows directly that these classes coincide respectively with the classes  $M_{D_1}$  ( $M_E$ ) and  $N_{D_1}$  ( $N_E$ ). Therefore a number of estimates obtained in the classes of functions studied in papers <sup>(2,3)</sup> are, respectively, estimates in the classes  $M_{D_1}$  ( $M_E$ ) and  $N_{D_1}$  ( $N_E$ ).

**Remark 3.** The results of the present note also hold in the case of  $n$  complex variables. All arguments are analogous, and in this case

$$L[F(z_1, \dots, z_n)] \equiv F(z_1, \dots, z_n) + \sum_{k=1}^n z_k F'_{z_k}(z_1, \dots, z_n).$$

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$$* (wF(w, 0))'_w \neq 0$$

in the disk  $D \cap \{z = 0\}$ , since condition (4) means that the function

$$w(wF(w, 0))'_w = w + \dots$$

regular in the disk  $D \cap \{z = 0\}$ , maps this disk univalently onto a domain star-shaped with respect to the origin <sup>(1)</sup>, and therefore in this disk has the unique simple zero  $w = 0$ .

*Note: Figure translations are in progress. See original paper for figures.*

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