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Abstract

Full Text

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ON THE INDEX OF SYSTEMS OF MULTIDIMENSIONAL SINGULAR INTEGRAL EQUATIONS

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A system of singular integral equations is considered:

$$a(x)u(x) + \int_S b(x, y - x)u(y)\, d_{yS} + Tu = f(x), \tag{1}$$

where S is a smooth n-dimensional manifold bounding a domain in $(n+1)$-dimensional Euclidean space, homeomorphic to a ball; $x = (x^1, \dots, x^{n+1})$, $y = (y^1, \dots, y^{n+1})$ are points on S; $a(x)$ is a complex square matrix of order p, defined and continuous on S; $b(x, \alpha)$ is a complex square matrix of order p, defined and continuous for $x \in S$ and arbitrary nonzero vectors $\alpha = (\alpha_1, \dots, \alpha_{n+1})$, with $b(x, \rho\alpha) = \rho^{-n} b(x, \alpha)$ $(\rho > 0,\ \alpha \neq 0)$; T is a regular integral operator. It is assumed that the condition for the existence of the singular integral in (1) is fulfilled (see ([1])).

The solution u of system (1) is sought in the space $\mathcal{L}^2(H)$ of functional columns of height p, whose elements are square-summable along S (satisfy a Hölder condition on S), and the right-hand side f is assumed to belong to the same spaces. S, a, and b are assumed sufficiently smooth. Under the stated assumptions, with the operator in (1) there is associated in a known way ([1]) a symbol $\Phi(\tau)$ –a square matrix of order p, defined and continuous on the set P of all unit vectors τ tangent to S; moreover, under multiplication of operators the symbols are multiplied. We shall assume that the condition

$$\det \Phi(\tau) \neq 0 \qquad (\tau \in P) \tag{2}$$

is fulfilled.

Then ([1]) system (1) is normally solvable, and the subspaces of solutions of the homogeneous system (1) $(f = 0)$ and of the homogeneous adjoint system are finite-dimensional. The difference $(\varkappa = k - k^*)$ between the dimensions k and k^* of these subspaces is called the **index** of system (1).

In ([3]), by a method based on topological and group-theoretic considerations, a formula was obtained for the index when $n = 2$, analogous to the known formula (see ([2])) for one-dimensional singular integral equations. Here, by the same method, the results of ([3]) are generalized.

Let L be the group of invertible complex square matrices of order p. To each continuous mapping $\Phi : P \to L$ there is assigned an integer $l(\Phi)$, defined as follows. For $p < n$, $l(\Phi) = 0$. For $p = n$, consider some row

$$(\Phi_1' + i\Phi_1'', \dots, \Phi_n' + i\Phi_n'')$$

of the matrix $\Phi(\tau)$. Let

$$\varphi(\tau) = (\Phi_1'(\tau), \Phi_1''(\tau), \dots, \Phi_n'(\tau), \Phi_n''(\tau)), \qquad \psi(\tau) = \frac{\varphi(\tau)}{|\varphi(\tau)|},$$

where $|\varphi|$ is the length of the vector φ. Then ψ maps P into the $(2n-1)$-dimensional unit sphere. By definition, $l(\Phi)$ is the normalized degree of this mapping. It is easy to see that $l(\Phi)$ does not depend on the arbitrary choice of the row of the matrix Φ. For $p > n$, as is known, the matrix $\Phi(\tau)$ can be continuously deformed, while preserving condition (2), into the matrix

$$\begin{pmatrix} E & 0 \\ 0 & \Phi_0(\tau) \end{pmatrix},$$

where E is the identity matrix of order $p - n$, and $\Phi_0(\tau)$ is a matrix of order n. The quantity $l(\Phi_0)$ does not depend on the method of deformation. By definition,

$$l(\Phi) = l(\Phi_0).$$

It follows from the definition that if Φ and Φ_1 are two mappings $P \to L$, then $l(\Phi\Phi_1) = l(\Phi) + l(\Phi_1)$, where $\Phi\Phi_1$ is the product of the matrices, and that if Φ and Φ_1 are homotopic, then $l(\Phi_1) = l(\Phi)$.

Lemma. *The index $\varkappa$ of system (1) is computed by the formula*

$$\varkappa = \gamma l(\Phi), \tag{3}$$

where Φ is the symbol of system (1), and γ is an integer constant.

Proof is carried out analogously to how this was done in ([4]) for $n = 2$. It suffices to restrict oneself to the case $p = n$, since the case $p > n$ is reduced to it by the deformation indicated above, and the case $p < n$ by passing to the matrix

$$\begin{pmatrix} E & 0 \\ 0 & \Phi \end{pmatrix},$$

which does not change the quantities $\varkappa$ and l. For fixed x, the matrix $\Phi(\tau(x))$ is homotopic to the identity. It follows from this that if $l(\Phi) = 0$, then Φ is homotopic to a matrix preserving constant values on tangent vectors issuing from one point. Therefore $\varkappa(\Phi) = 0$, which proves (3) (see also ([4])).

Theorem. *For even n, the index $\varkappa$ of system (1) is computed by formula (3), in which $\gamma = 1$ (for the corresponding orientation of P).*

Proof. As in ([3]), one considers the boundary-value problem for a system of harmonic functions associated with a system of first-order equations on the sphere. In constructing an elliptic first-order system on the sphere, the results of A. A. Dezin ([5]) are used: one considers the system (k_n) (([5]), p. 28) for even n, having index 2. This system is transformed into an elliptic system (with complex coefficients), acting no longer in spaces of covariants but in spaces of vectors whose elements are absolute scalars. Constructively, such a transformation can be carried out as follows. A special coordinate system is introduced (for $n = 2$ see ([4]); in the general case it is analogous). The Jacobi matrix M on the equator is deformed into a constant one:

$$M(1-t) + itE$$

$(0 \leq t \leq 1)$, and therefore can be continued to the lower hemisphere. One proceeds analogously for all associated matrices. With the aid of the matrices obtained, a linear change of the unknown functions and right-hand sides is made in the system (k_n). The elliptic system obtained in this way from the system (k_n) has, obviously, index 2. Further, as in ([3]), one obtains a system of the form (1) with index 2 and symbol $\Phi(\tau) = B^j(x)\tau_j$. To prove that $l(\Phi)$ is even, put $\Phi_1(\tau) = C^j\tau_j$, where C^j $(j = 1, \dots, n+1)$ are complex square matrices of order 2^{n-1} such that $\det C^j\alpha_j \neq 0$ $(\alpha \neq 0)$. As is known, such matrices exist. Since C^j do not depend on x, $l(\Phi_1) = 0$. But $\Phi(\tau)\Phi_1(\tau)$ does not change under the replacement of τ by $-\tau$, and therefore $l(\Phi) = l(\Phi\Phi_1)$ is an even number.

As a consequence of the theorem one obtains a formula for the index of elliptic systems of differential equations on an even-dimensional sphere. The characteristic matrix of such a system of order m in local coordinates has the form

$$A(\xi,\alpha) = A^{i_1\cdots i_m}(\xi)\alpha_{i_1}\dots\alpha_{i_m}, \tag{4}$$

where $A^{i_1\cdots i_m}$ is a matrix whose elements are contravariant tensors, and α_i is a covariant vector. With the aid of (4) a mapping $A : P \to L$ is defined (the ellipticity condition). The formula for the index $\varkappa$ of an elliptic system with characteristic matrix (4) has the form $\varkappa = l(A)$.

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Note: Figure translations are in progress. See original paper for figures.