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## Abstract

## Full Text

## MECHANICS

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# ON SOME CASES OF INTEGRABILITY OF THE EQUATIONS OF MOTION OF A GYROSTAT

1. Let us consider a heavy rigid body fixed at one point. Suppose that with this body there is associated a certain axis about which a homogeneous flywheel can rotate. Using the usual notation in rigid-body dynamics and denoting by $\lambda, \mu, \nu$ the projections of the angular momentum of the flywheel on the moving axes, we write the equations of motion of the rigid body under consideration (the gyrostat) in the following form:

$$A\frac{dp}{dt} + (C-B)qr + (\nu q - \mu r) = Mg(\zeta\gamma' - \eta\gamma''),$$

$$B\frac{dq}{dt} + (A-C)rp + (\lambda r - \nu p) = Mg(\xi\gamma'' - \zeta\gamma), \tag{1}$$

$$C\frac{dr}{dt} + (B-A)pq + (\mu p - \lambda q) = Mg(\eta\gamma - \xi\gamma');$$

$$\frac{d\gamma}{dt} + q\gamma'' - r\gamma' = 0,$$

$$\frac{d\gamma'}{dt} + r\gamma - p\gamma'' = 0, \tag{2}$$

$$\frac{d\gamma''}{dt} + p\gamma' - q\gamma = 0.$$

In the present paper we intend to indicate two particular cases of integrability of these equations. The first case is connected with the Goryachev-Chaplygin gyroscope ($A = B = 4C,\ \eta = \zeta = 0,\ \xi \neq 0$), the second with Appelrot' s case: $\eta = 0,\ A(B-C)\xi^2 - C(A-B)\zeta^2 = 0$.

2. For the Goryachev-Chaplygin gyroscope, equations (1) are written as

$$4\frac{dp}{dt} - 3qr + (\nu q - \mu r) = 0,$$

$$4\frac{dq}{dt} + 3rp + (\lambda r - \nu p) = a\gamma'', \tag{3}$$

$$\frac{dr}{dt} + (\mu p - \lambda q) = -a\gamma',$$

where $\lambda, \mu, \nu$ denote the former quantities $\lambda, \mu, \nu$, but divided by $C$, and

$$a = \frac{Mg}{C}\xi.$$

We shall show that if $\lambda$ and $\mu$ are equal to zero, then equations (3) will have, in addition to the energy integral

$$4(p^2 + q^2) + r^2 + 2a\gamma = h,$$

one more integral, depending on an arbitrary constant, if the constant of areas $m$ in the integral of angular momentum is equal to

zero:

$$4(pr + qr') + rr'' + \nu r'' = 0. \tag{4}$$

This new integral will be

$$(r - \nu)(p^2 + q^2) - apr'' = k, \tag{5}$$

where $k$ is an arbitrary constant number. To verify this assertion, denote by $S$ the left-hand side of the preceding equality.

Using equations (1) and (2), it is easy to find that

$$\frac{dS}{dt} = -\frac{1}{4}aq\,[4(pr + qr') + (r + \nu)r''],$$

but, by virtue of equality (4), which holds throughout the motion, the right-hand side of this equation vanishes, and we obtain the integral (5) with an arbitrary constant $k$.

Denote by $u$ and $v$ two auxiliary functions of time, and introduce into consideration the following two polynomials of the third degree:

$$U = u(u-\nu)^2 - hu - 4k,$$
$$V = v(v+\nu)^2 - hv + 4k.$$

With the aid of these polynomials the sought unknowns $p, q, r, \gamma, \gamma', \gamma''$ are found from formulas generalizing Chaplygin’ s formulas ($^1$):

$$8ap = \sqrt{(U+2au)(2av-V)} - \sqrt{(2au-U)(V+2av)},$$

$$8aq = \sqrt{(U+2au)(V+2av)} + \sqrt{(2au-U)(2av-V)},$$

$$r = u - v - \nu,$$

$$2a\gamma = h - (u-v-\nu)^2 - uv,$$

$$2a\gamma' = -\frac{\sqrt{4a^2u^2-U^2} - \sqrt{4a^2v^2-V^2}}{u+v},$$

$$2a\gamma'' = \frac{\sqrt{(U+2au)(2av-V)} + \sqrt{(2au-U)(V+2av)}}{u+v}.$$

The functions $u, v$ are found by integrating the equations:

$$\frac{du}{\sqrt{4a^2u^2-U^2}} - \frac{dv}{\sqrt{4a^2v^2-V^2}} = 0, \qquad \frac{2u\,du}{\sqrt{4a^2u^2-U^2}} + \frac{2v\,dv}{\sqrt{4a^2v^2-V^2}} = dt.$$

From these equations the unknown functions $u, v$ will be found in terms of theta functions of two variables.

3. Let us return to the general equations (1) and (2) and set in them $\eta = 0$, $\mu = 0$. Then the system of equations (1) is written as follows:

$$A\frac{dp}{dt} + (C-B)qr + \nu q = Mg\,\zeta\gamma',$$

$$B\frac{dq}{dt} + (A-C)rp + (\lambda r - \nu p) = Mg\,(\xi\gamma'' - \zeta\gamma),$$

$$C\frac{dr}{dt} + (B-A)pq - \lambda q = -Mg\,\xi\gamma'.$$

Eliminating the unknown $\gamma'$ from the two extreme equations, we obtain:

$$\frac{d}{dt}(A\xi p + C\zeta r) + q\big[(B-A)\zeta p + (C-B)\xi r + (\nu\xi - \lambda\zeta)\big] = 0.$$

It follows from this that if between the principal moments of inertia and the coordinates of the center of gravity of the gyrostat there exists the Appellrot condition ($^2$)

$$A(B-C)\xi^2 - C(A-B)\zeta^2 = 0,$$

then the equations of motion will have the following particular integral:

$$(A-B)\xi p + (B-C)\zeta r + (\lambda\zeta - \nu\xi) = 0.$$

Adjoining to this integral two other integrals,

$$\frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + Mg(\xi\gamma + \zeta\gamma'') = h,$$

$$(Ap+\lambda)\gamma + Bq\gamma' + (Cr+\nu)\gamma'' = k,$$

we obtain the possibility of reducing the solution of the problem of the motion of a gyrostat to the integration of a Riccati equation.

From the point of support describe a sphere of unit radius and denote by $V$ the point of intersection of the upward vertical with the surface of the sphere; by $a, b, c$ denote the points of intersection of the principal axes of inertia of the body with the sphere. On the arc of the great circle $ac$ will lie the point $G$ at which the surface of the sphere meets the straight line going from the point of support to the center of gravity; denote the arc $aG$ by $\alpha$, and note that $\xi = l\cos\alpha,\ \zeta = l\sin\alpha$, where $l$ is the distance from the point of support to the center of gravity. Denote by $\theta$ the arc $VG$, and by $\varphi$ the angle between the continuation of the arc $VG$ and the arc of the great circle $Gb$. Next take on the sphere some great circle $VK$ as initial, and measure from it the angular distance $\psi$ between the arc $VK$ and the moving arc $VG$ $^{(3)}$. With these notations we shall have the following differential equations:

$$(1-u^2)\frac{d\psi}{dt} = k + \left(\frac{\nu\sin\alpha}{C-B} - \frac{\lambda\cos\alpha}{B-A}\right)u,$$

$$\left(\frac{du}{dt}\right)^2 = \left(h - \frac{2Mgl}{B}u\right)(1-u^2) - \left[k + \left(\frac{\nu\sin\alpha}{C-B} - \frac{\lambda\cos\alpha}{B-A}\right)u\right]^2,$$

$$\frac{d\varphi}{dt} + u\frac{d\psi}{dt} = D + E\left(\sin\varphi\,\frac{d\psi}{dt}\sqrt{1-u^2} - \frac{\cos\varphi}{\sqrt{1-u^2}}\,\frac{du}{dt}\right),$$

where

$$u = \cos\theta, \qquad D = \frac{2(\nu\cos\alpha - \lambda\sin\alpha)}{(A-C)\sin 2\alpha}, \qquad E = \frac{2(A-B)(B-C)}{B(A-C)\sin 2\alpha}.$$

The first two equations show that the straight line going from the point of support to the center of gravity of the gyrostat moves in space exactly as the axis of a Lagrange gyroscope. The last equation gives the angle of proper rotation $\varphi$; it can be reduced to a Riccati equation and then to a linear differential equation of the second order with doubly periodic coefficients.

In the special case in which the relation $\nu\xi - \lambda\zeta = 0$ holds, the motion of the gyrostat can be described just as vividly as the motion of Hess' s loxodromic pendulum (4).

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*