INVARIANT CONSTRUCTION OF THE DIFFERENTIAL GEOMETRY OF A MANIFOLD OF PLANE ALGEBRAIC ELEMENTS

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Abstract

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V. S. MALAKHOVSKII

INVARIANT CONSTRUCTION OF THE DIFFERENTIAL GEOMETRY OF A MANIFOLD OF PLANE ALGEBRAIC ELEMENTS

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In this paper an m-dimensional manifold V_m^k is studied, whose element is a nondegenerate algebraic hypersurface of even order $k = 2p$ of a hyperplane of the n-dimensional projective space P_n (a plane algebraic element of order k). A fundamental object has been obtained which determines the manifold up to its position in the space. A number of objects invariantly associated with the manifold are found, and their geometric characterization is clarified.

1. If the vertices A_α $(\alpha, \beta, \sigma_q, \dots = 1, 2, \dots, n)$ of a moving frame are placed in the hyperplane of the algebraic element, and the vertex A_{n+1} outside it, then the equations of the plane algebraic element are written in the form

$$a^1_{x_1 \dots \alpha_k} x^{\alpha_1} \cdots x^{\alpha_k} = 0, \qquad x^{n+1} = 0, \tag{1}$$

where the hyperdeterminant a of the symmetric spatial matrix $(a_{\alpha_1 \dots \alpha_k})$ may always be assumed equal to unity $(^1)$. The system of differential equations of invariance $(^2)$ of the algebraic element (1) has the form

$$\omega_\alpha = 0, \qquad \nabla a_{\alpha_1 \dots \alpha_k} + \frac{k}{n} a_{\alpha_1 \dots \alpha_k} \omega_\beta^\beta = 0, \tag{2}$$

where ∇ is the symbol of covariant differentiation, $\omega_\alpha = \omega_\alpha^{n+1}$, and $\omega_{\alpha'}^{\beta'}$ $(\alpha', \beta', \dots = 1, \dots, n+1)$ are the components of the derivative formulas $d\bar{A}_{\alpha'} = \omega_{\alpha'}^{\beta'} \bar{A}_{\beta'}$ of the moving frame. System (2) determines a $(C_{n+k-1}^k + n - 1)$-dimensional space of plane algebraic elements of order k. If the dimension m of the manifold V_m^k exceeds the number h of parameters on which the hyperplanes containing the algebraic elements in it depend, i.e. $m = h + p$, then the manifold V_m^k is considered on ∞^h submanifolds with a fixed hyperplane. Therefore we shall restrict ourselves to consideration of the case $m = h$. The closed system of differential equations

$$\Delta a^j_{\check{i}} = a^j_{\check{i}} \omega_j, \qquad \nabla a_{\alpha_1 \dots \alpha_k} + \frac{k}{n} a_{\alpha_1 \dots \alpha_k} \omega_\beta^\beta = b^j_{\alpha_1 \dots \alpha_k} \omega_j,$$

$$[\Delta a^{j}_{\check{i}}\omega_{j}] = 0, \qquad [\Delta b^{j}_{\alpha_1\dots\alpha_k}\omega_{j}] = 0 \tag{3}$$

determines a manifold V^k_m of this type with an arbitrariness $C^k_{n+k-1}+n-m-1$ functions of m arguments. Here $i, j, \dots = 1, \dots, m$;

$$\check{i}, \check{j}, \dots = m+1, \dots, n; \qquad \Delta a^{j}_{\check{i}} = \nabla a^{j}_{\check{i}} + a^{i}_{\check{i}} a^{i}_{j} \omega^{j}_{\check{i}} - \omega^{j}_{\check{i}},$$

$$\Delta b^{j}_{\alpha_1\dots\alpha_k} = \nabla b^{j}_{\alpha_1\dots\alpha_k} + b^{j}_{\alpha_1\dots\alpha_k}\left(\frac{k}{n}\omega^{\beta}_{\beta} - \omega^{n+1}_{n+1}\right) + a^{i}_{j} b^{j}_{\alpha_1\dots\alpha_k}\omega^{j}_{i} +$$

$$+\frac{k}{n} a_{\alpha_1\dots\alpha_i}\left(\omega^{j}_{n+1} + a^{j}_{\check{i}}\omega^{\check{i}}_{n+1}\right) - \{a_{\beta\alpha_2\dots\alpha_k}(\delta^{j}_{\alpha_1} + \delta^{\check{i}}_{\alpha_1} a^{j}_{\check{i}}) + \cdots$$

$$\cdots + a_{\alpha_1\dots\alpha_{k-1}\beta}(\delta^{j}_{\alpha_k} + \delta^{\check{i}}_{\alpha_k} a^{j}_{\check{i}})\}\omega^{\beta}_{n+1},$$

where the quantities $b^i_{\alpha_1\dots\alpha_k}$, $\Delta b^i_{\alpha_1\dots\alpha_k}$ are symmetric with respect to any pair of lower indices.

indices and satisfy the identities

$$a^{\alpha_1\cdots\alpha_k} b^{i}_{\alpha_1\dots\alpha_k} = 0, \qquad a^{\alpha_1\cdots\alpha_k}\,[\Delta b^{i}_{\alpha_1\dots\alpha_k}\omega] = 0,$$

where $a^{\alpha_1\cdots\alpha_k}$ are the algebraic complements of the elements $a_{\alpha_1\dots\alpha_k}$ of the hyperdeterminant a.

From (3) it follows:

Theorem 1. *The intrinsic fundamental object*

$$\Gamma = \{a^{j}_{i},\ da_{\alpha_1\dots\alpha_k},\ b^{i}_{\alpha_1\dots\alpha_k}\}$$

is the fundamental object of the manifold V^k_m. The proper specification of the components of its first prolongation determines the manifold V^k_m up to constants (^{2}).

The systems of quantities a^j_i and $a_{\alpha_1\dots\alpha_k}$ form subobjects of the object Γ, and $a_{\alpha_1\dots\alpha_k}$ is a symmetric tensor. If $m < n$, then the geometric object a^j_i determines in the hyperplane of the locally algebraic element of the manifold V^k_m an $(n-m-1)$-dimensional characteristic subspace

$$x^{i} + x^{i} a^{j}_{i} = 0, \qquad x^{n+1} = 0, \tag{4}$$

consisting of points which, under transition to a neighboring algebraic element, remain in the same hyperplane $x^{n+1} = 0$. On the manifold V^2_m the system of quantities $a^{\alpha\beta}$ determines a twice contravariant symmetric tensor.

2. Consider the manifold V_n^k. The system of quantities $(a_{\alpha_1\cdots\alpha_k}, b_{\alpha_1\cdots\alpha_{k-1}})$, where $b_{\alpha_1\cdots\alpha_{k-1}} = b^{\beta}_{\alpha_1\cdots\alpha_{k-1}\beta}$, determines a linear homogeneous object enveloping the tensor $a_{\alpha_1\cdots\alpha_k}$. For manifolds V_n^2 this object has a simple geometric characteristic. It determines in the space P_n an invariant pencil of hyperquadrics

$$a_{\alpha\beta}x^{\alpha}x^{\beta} + \frac{2n}{(n-1)(n+2)}\, b_{\alpha}x^{\alpha}x^{n+1} + \lambda(x^{n+1})^2 = 0, \tag{5}$$

containing the local quadratic element. The quasitensor $b^{\alpha} = a^{\alpha\beta}b_{\beta}$ determines the invariant point

$$\overline{B} = b^{\alpha}\overline{A}_{\alpha} + \frac{2-n(n+1)}{n}\,\overline{A}_{n+1},$$

which is the vertex of the hypercone of the pencil (5).

The system of quantities

$$b_{\alpha_1\cdots\alpha_{2k-1}} = a_{\beta(\alpha_1\cdots\alpha_{k-1}}b^{\beta}_{\alpha_k\cdots\alpha_{2k-1})} - \frac{k}{n+k}\, b_{(\alpha_1\cdots\alpha_{k-1}}a_{\alpha_k\cdots\alpha_{2k-1})}, \tag{6}$$

where the parentheses denote cyclic alternation, forms a $(2k-1)$-times covariant symmetric tensor determining, in the hyperplane of each locally algebraic element, an invariant $(n-2)$-dimensional algebraic surface of order $2k-1$ (the attached surface)

$$b_{\alpha_1\cdots\alpha_{2k-1}}x^{\alpha_1}\cdots x^{\alpha_{2k-1}} = 0, \qquad x^{n+1} = 0.$$

The common points of the locally algebraic element (1) and of the attached surface form the t-focal manifold of the algebraic element. For $n = 3$ this manifold consists of $k(2k-1)$ t-focal points of a plane algebraic curve, which have a simple geometric characteristic.

Theorem 2. *In order that a point M of the curve (1) be a t-focal point of this curve, it is necessary and sufficient that it be a focus of the nonholonomic congruences (curves) determined by all points of the tangent to the curve (1) at M.*

Remark. If, in the plane $x^4 = 0$ of the curve (1), one takes an arbitrary point $N\,(q^{\alpha}, 0)$ $(\alpha = 1, 2, 3)$, then all displacements of this point lying in the plane of the curve satisfy the equation $q^{\alpha}\omega_{\alpha} = 0$, which determines a nonholonomic congruence of curves belonging to the complex V_3^k. Consequently, each point in the plane of the curve uniquely determines a nonholonomic curvilinear congruence.

3. On the manifold V_n^2 of quadratic elements there are defined the tensors $b_{\alpha\beta} = a^{\mu\eta}a^{\nu\varkappa}b_{\mu\nu}b_{\eta\varkappa\beta}$ and $c^{\alpha} = a^{\alpha\mu}a^{\beta\nu}a^{\gamma\eta}b_{\mu\nu}b_{\beta\gamma}$, and the following system of relative and absolute invariants:

$$b_0 = a^{\alpha\beta} b_{\alpha\beta}, \quad c_0 = a_{\alpha\beta} c^\alpha c^\beta, \quad \hat{c}_0 = b_{\alpha\beta} c^\alpha c^\beta, \quad \hat{c} = b_{\alpha\beta\gamma} c^\alpha c^\beta c^\gamma, \quad b = \det(b_{\alpha\beta}); \tag{7}$$

$$B_0 = \frac{b_0^n}{b}, \quad C_0 = \frac{c_0^n}{b^3}, \quad \hat{C}_0 = \frac{\hat{c}_0^n}{b^4}, \quad \hat{C} = \frac{\hat{c}^n}{b^5}. \tag{8}$$

For such a manifold the following "principle of transfer" is valid.

Theorem 3. *The differential geometry of an n-dimensional manifold V_n^2 of quadratic elements of the space P_n may be regarded as the geometry of a certain regular hypersurface of the $(n+1)$-dimensional tangential centroprojective space P_0^{n+1}, in which the original n-dimensional point space P_n plays the role of a fixed point.*

4. Let us consider a manifold V_m^2 of dimension $m < n$. In the general case the characteristic subspace (4) does not intersect its polar subspace, and we can place the vertices $A_{\breve{i}}$ of the frame in the subspace (4), and the vertices A_i in the polar subspace. Then $a_{\breve{i}}^{i} = 0$, $a_{\breve{i}\breve{j}} = 0$, and the Pfaff forms $\omega_{\breve{i}}^{i}$, $\omega_{i}^{\breve{i}}$ become the principal forms of the manifold V_m^2, and we may put $\omega_{\breve{i}}^{j} = b_{\breve{i}}^{jp}\omega_p$, $\omega_i^{\breve{i}} = a^{\breve{i}j}(b_{\breve{i}j}^{p} + a_{iq} b_{\breve{j}}^{qp})\omega_p$.

On the manifold V_m^2 there exist the following geometric objects:

1) The tensors a_{ij}, $a_{\breve{i}\breve{j}}$, a^{ij}, $a^{\breve{i}\breve{j}}$. The equations $x^{n+1} = 0$, $a_{ij}x^i x^j = 0$ and $x^{n+1} = 0$, $a_{\breve{i}\breve{j}} x^{\breve{i}} x^{\breve{j}} = 0$ define invariant hypercones whose vertices are, respectively, the characteristic subspace and the polar subspace.

2) The quasitensor $\hat{a}^i = a^{jp} b_{jp}^i$, which determines the $(n-m)$-dimensional invariant plane $x^i + \frac{n}{2(n-m)}\hat{a}^i x^{n+1} = 0$.

3) The quasitensors $\hat{b}^i = a^{ij} b_{jp}^p$, $\hat{b}^{\breve{i}} = a^{\breve{i}\breve{j}} b_{p\breve{j}}^{p}$, which determine the invariant point

$$\bar{B} = \bar{A}_{n+1} - \frac{n}{n(m+1)-2}\hat{b}^i \bar{A}_i - \frac{1}{m}\hat{b}^{\breve{i}} \bar{A}_{\breve{i}}$$

of the space P_n.

4) The vector

$$a^i = \hat{a}^i - \frac{2(n-m)}{n(m+1)-2}\hat{b}^i,$$

which determines in the polar subspace the invariant point $\bar{A} = a^i \bar{A}_i$.

5) The thrice covariant tensor

$$b_{ijp} = a_{q(i} b_{jp)}^{q} - \frac{2(n-1)}{n(m+2)-1}\hat{b}^q a_{q(i} a_{jp)}.$$

It determines the associated t-focal manifold

$$a^{ij}x^i x^j = 0, \quad b_{ijp}x^i x^j x^p = 0, \quad x^{n+1} = 0,$$

consisting of an $(m-3)$-parameter family of $(n-m)$-dimensional plane generators passing through the characteristic subspace (4). For $m = 3$ the polar subspace (plane) contains, in the general case, 6 t-focal points belonging to the invariant conic

$$a_{ij}x^i x^j = 0, \quad x^{\breve{i}} = 0, \quad x^{n+1} = 0 \tag{9}$$

and possessing a simple geometric characteristic.

Theorem 4. *In order that a point M of the conic $a_{ij}x^i x^j = 0$, $x^{n+1} = 0$ of the manifold V_3^2 be t-focal, it is necessary and sufficient that it be projected from the characteristic subspace to a point M^* of the conic (9) that is the focus of all nonholonomic congruences (cones) determined by the points of the tangent to the conic (9) at M^*.*

Tomsk State University
named after V. V. Kuibyshev

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Note: Figure translations are in progress. See original paper for figures.

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