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Abstract

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ON THE VIETORIS-BEGLE THEOREM

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The classical Vietoris-Begle theorem ([2,3]) asserts that if, under a continuous mapping of a bicompactum X onto a bicompactum Y, the inverse images of points are acyclic in dimensions $0, 1, \dots, N-1$, then the induced mappings of cohomology $H^i(Y) \to H^i(X)$ are isomorphisms for $i \leq N-1$, and the mapping $H^N(Y) \to H^N(X)$ is monomorphic. In the present paper the behavior of cohomology under a continuous mapping is studied in the case when the inverse images of points are not necessarily acyclic, but non-acyclicity occurs only on sets of sufficiently small dimension.

Let M be a subset in a topological space X. Following P. S. Aleksandrov ([1]), we shall call the relative dimension $\operatorname{rd}_X M$ of the set M in X the greatest of the dimensions $\dim E$ of the sets E closed in X and contained in M. If the set M is empty, we shall assume that $\operatorname{rd}_X M = -\infty$.

Theorem 1. *Let f be a closed mapping of a paracompactum X onto a paracompactum Y, L a sheaf of abelian groups over Y, L^* the inverse image of the sheaf L over X. Let*

$$M_0 = \{y \in Y \mid H^0(f^{-1}y; L^*) \neq L_y\},$$

$$M_k = \{y \in Y \mid H^k(f^{-1}y; L^*) \neq 0\}, \qquad k \geq 1 \; **.$$

Let $d_k = \operatorname{rd}_Y M_k$; put

$$n = 1 + \max_{0 \leq k \leq N-1} (d_k + k),$$

*where N is some natural number or $+\infty$. If $n < N$, then the mapping $H^i(Y; L) \to H^i(X; L^*)$ is epimorphic for $i = n$, isomorphic for $n < i < N$, and monomorphic for $i = N$ ***.*

In the case when all the sets M_k are empty and L is a constant sheaf, Theorem 1 turns into the Vietoris-Begle theorem. If, as L, one takes a sheaf that is constant on some open set $U \subset Y$ and equal to zero on $Y \setminus U$, then we obtain the

Vietoris–Begle theorem for relative cohomology. Let us note some consequences of Theorem 1.

Corollary 1. *Let f be a closed mapping of a paracompactum X onto a paracompactum Y, L a sheaf of abelian groups over Y, L^* the inverse* is an image of the sheaf L over X. Let A_0 be the set of points $y \in Y$ whose full preimages are disconnected,

$$A_k = \{y \in Y \mid \dim f^{-1}y \geq k\}, \qquad k = 1, 2, \ldots$$

Let $d_k = \operatorname{rd}_Y A_k$; put

$$n = 1 + \max_{k \geq 0}(d_k + k).$$

Then the map $H^n(Y; L) \to H^n(X; L^*)$ is an epimorphism, and the maps $H^i(Y; L) \to H^i(X; L^*)$, $i > n$, are isomorphisms.

In particular, if the map f is zero-dimensional, then $n = 1 + d_0$, and, consequently, the map $H^{1+d_0}(Y; L) \to H^{1+d_0}(X; L^*)$ is an epimorphism. The following special case of this assertion plays an essential role in the theory of bicompact extensions of peripherally bicompact spaces.

Let Y_1 and Y_2 be bicompact extensions of a completely regular space X with punctiform* remainders $Y_1 \setminus X$ and $Y_2 \setminus X$, and suppose that the extension Y_2 follows the extension Y_1, i.e. there exists a map $f : Y_2 \to Y_1$ that is the identity on the points of the space X. Then the map $H^1(Y_1; G) \to H^1(Y_2; G)$ is an epimorphism** (G is an arbitrary coefficient group).

Under the assumption that the remainders are closed, this assertion was proved in (^{7}).

Corollary 2. Let f be a closed mapping of a paracompactum X onto a paracompactum Y, increasing dimension, i.e. $\dim Y > \dim X$. Let A_0 be the set of points $y \in Y$ whose full preimages are disconnected,

$$A_k = \{y \in Y \mid \dim f^{-1}y \geq k\}, \qquad k = 1, 2, \ldots$$

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* A paracompactum is a paracompact Hausdorff space.

** L_y denotes the stalk of the sheaf L over the point $y \in Y$. A sheaf over the whole space and the sheaves induced by it on subsets are denoted by one and the same symbol.

*** In particular, as L one may take an arbitrary group of coefficients G; in this case $G^* = G$, and M_k is the set of those points in Y whose full inverse images are non-acyclic over G in dimension k.

Let $d_k = \mathrm{rd}_Y\, A_k$. If $\dim Y = m < \infty$ and $d_k + k + 1 < m$, $k = 1, 2, \ldots$, then $\mathrm{rd}_Y\, M_0 \geq m - 1$.

In the case when the map f is zero-dimensional, A_0 coincides with the set of those points of the space Y whose preimages consist of more than one point, while the sets A_k for $k \geq 1$ are empty. Therefore Corollary 2 becomes the well-known theorem of V. Gurevich (^{5}).

Corollary 3. Let f be a closed mapping of a paracompactum X onto a paracompactum Y; let M_0 be the set of points $y \in Y$ whose full preimages are disconnected,

$$M_k = \{y \in Y \mid H^k(f^{-1}y; Z) \neq 0\}, \qquad k = 1, 2, \ldots$$

Let $d_k = \mathrm{rd}_Y\, M_k$. If $\dim Y = m < \infty$ and $1 + d_k + k < m$, $k = 0, 1, 2, \ldots$, then $\dim Y \leq \dim X$, i.e. the map f does not increase dimension.

In the case when all the sets M_k are empty, we obtain the well-known theorem of Dyer that a map for which the preimages of all points are acyclic does not increase dimension (^{6}).

In the author' s paper (^{8}) the following proposition was proved (although formulated only for a special case), which may be regarded as the converse of the Vietoris–Begle theorem for $N = 1$.

Let f be a closed mapping of a paracompactum X onto a paracompactum Y such that the full preimages of all points $y \in Y \setminus M$ are connected, where M is some subset of Y of relative dimension zero. If the induced map $H^0(Y; Z) \to H^0(X; Z)$ is an isomorphism, and the map $H^1(Y; Z) \to H^1(X; Z)$ is a monomorphism, then the map f is monotone, i.e. the full preimages of all points $y \in Y$ are connected.

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* A space is called punctiform if every connected bicompact subset of it consists of one point.

** We note that the kernel of this epimorphism is a torsion-free group if G is a torsion-free group.

An analogous converse is also admitted by Theorem 1.

Theorem 2. *Let f be a closed mapping of a paracompactum X onto a paracompactum Y, L a sheaf of Abelian groups over Y, and L^* the inverse image of the sheaf L over X. Let*

$$M_0 = \{y \in Y \mid H^0(f^{-1}y; L^*) \neq L_y\},$$

$$M_k = \{y \in Y \mid H^k(f^{-1}y; L^*) \neq 0\}, \qquad k \geqslant 1,$$

and $d_k = \mathrm{rd}_Y\, M_k$. *Suppose that*

$$1 + \max_{\substack{d_k > 0 \\ 0 \le k \le N-1}} (d_k + k) < N,$$

where N *is some natural number or* $+\infty$. *Let* n *be an integer such that*

$$1 + \max_{\substack{d_k > 0 \\ 0 \le k \le N-1}} (d_k + k) \leqslant n < N,$$

or, if all $d_k \leqslant 0$ *for* $k \leqslant N-1$, *then* $0 \leqslant n < N$. *If the mapping* $H^i(Y; L) \to H^i(X; L^*)$ *is epimorphic for* $i = n$, *isomorphic for* $n < i < N$, *and monomorphic for* $i = N$, *then for all points* $y \in Y$ *we have* $H^i(f^{-1}y; L^*) = 0$ *for* $n \leqslant i < N$, $i > 0$; *if* $n = 0$, *then, in addition, for all points* $y \in Y$ *we have* $H^0(f^{-1}y; L^*) = L_y$.

The proposition formulated above from (8) follows from Theorem 2 for $n = 0$ and $N = 1$. If in this proposition one additionally requires that the mapping f be zero-dimensional, then we obtain that the mapping f is homeomorphic. In particular, the following assertion for bicompact extensions is obtained.

Let Y_1 and Y_2 be bicompact extensions of a completely regular space X with pointlike remainders, with the extension Y_2 following the extension Y_1; let $f : Y_2 \to Y_1$ be the corresponding mapping. If the mapping $H^1(Y_1; Z) \to H^1(Y_2; Z)$ is monomorphic (and, consequently, isomorphic), and the mapping $H^0(Y_1; Z) \to H^0(Y_2; Z)$ is isomorphic, then the extensions Y_1 and Y_2 coincide, i.e. f is a homeomorphism.

For the case of closed remainders this assertion was proved in (7).

Białynicki-Birula in (4) generalizes the Vietoris–Begle theorem in another direction. He considers a triple of spaces X, Y, T and mappings $f : X \to Y$, $g : Y \to T$, $h = gf : X \to T$, and proves that the assertion of the Vietoris-Begle theorem remains valid if, instead of acyclicity of the inverse images of points under the mapping f, one requires that, for every point $t \in T$, the mappings $H^i(g^{-1}t) \to H^i(h^{-1}t)$, $i = 0, 1, \dots, N-1$, be isomorphic, and the mapping $H^N(g^{-1}t) \to H^N(h^{-1}t)$ be monomorphic. Białynicki-Birula also proves the converse of this theorem. The following two theorems include both Theorems 1 and 2 and the theorems of Białynicki-Birula.

Theorem 3. *Let* X, Y, T *be paracompacta,* $f : X \to Y$, $g : Y \to T$ *closed superjective* mappings, and* $h = gf : X \to T$. *Let* L *be a sheaf of Abelian groups over* Y, *and* L^* *the inverse image of the sheaf* L *over* X. *Let* M_k *be the set of points* $t \in T$ *for which the mapping*

$$H^k(g^{-1}t; L) \to H^k(h^{-1}t; L^*)$$

is not an isomorphism, and let $d_k = \mathrm{rd}_T\, M_k$. *Put*

$$n = 1 + \max_{0 \leqslant k \leqslant N-1} (d_k + k),$$

where N is some natural number or $+\infty$. If $n < N$, then the mapping $f^* : H^i(Y; L) \to H^i(X; L^*)$ is epimorphic for $i = n$ and isomorphic for $n < i < N$. If, moreover, for every point $t \in T$ the mapping $H^N(g^{-1}t; L) \to H^N(h^{-1}t; L^*)$ is monomorphic, then the mapping $f^* : H^N(Y; L) \to H^N(X; L^*)$ is also monomorphic.

Theorem 4. Let X, Y, T be paracompacts, $f : X \to Y$, $g : Y \to T$ closed surjective mappings, and $h = gf : X \to T$. Let L be a sheaf of Abelian groups over Y, and let L^* be the inverse image of the sheaf L over X. Let M_k be the set of points $t \in T$ for which the mapping $H^k(g^{-1}t; L) \to H^k(h^{-1}t; L^*)$ is not an isomorphism, and let $d_k = \mathrm{rd}_T\, M_k$. Suppose that

$$1 + \max_{\substack{d_k > 0 \\ 0 \leq k \leq N-1}} (d_k + k) < N,$$

where N is some natural number or $+\infty$. Let n be such an integer that

$$1 + \max_{\substack{d_k > 0 \\ 0 \leq k \leq N-1}} (d_k + k) \leq n < N,$$

or, if all $d_k \leq 0$ for $k \leq N - 1$, then $0 \leq n < N$. Suppose that the mapping $H^i(Y; L) \to H^i(X; L^*)$ is epimorphic for $i = n$, isomorphic for $n < i < N$, and monomorphic for $i = N$. Then for every point $t \in T$ the mapping $H^i(g^{-1}t; L) \to H^i(h^{-1}t; L^*)$ is epimorphic for $i = n$ and isomorphic for $n < i < N$. If, moreover, the set of points for which the mapping $H^N(g^{-1}t; L) \to H^N(h^{-1}t; L^*)$ is not monomorphic has relative dimension ≤ 0, then this mapping is monomorphic for all points $t \in T$.

If $Y = T$ and g is the identity mapping, then Theorems 3 and 4 become, respectively, Theorems 1 and 2. If in Theorem 3 all the sets M_k are empty for $0 \leq k \leq N - 1$, then this theorem becomes the first theorem of Bialynicki-Birula. The essential difference between Theorem 4 and the second theorem of Bialynicki-Birula consists in the fact that in his theorem it is additionally assumed that all the sets M_k for $0 \leq k \leq N - 1$ are contained in one and the same set T_1 of relative dimension zero and that for every point $t \in T \setminus T_1$ the groups $H^i(g^{-1}t)$ and $H^i(h^{-1}t)$, $0 < i < N$, are not only isomorphic but also equal to zero.

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[*] A mapping $\alpha : A \to B$ is called superjective if $\alpha(A) = B$.

The proofs of Theorems 1–4 are based on the Leray spectral sequence.

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