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Abstract

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SEMISIMPLE DECOMPOSITIONS OF SEMISIMPLE LIE ALGEBRAS

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Let G be a Lie algebra, and let G' and G'' be its subalgebras. We say that the triple (G, G', G'') is a **decomposition** if $G = G' + G''$. A Lie group acting on a manifold M is called **locally transitive** if at least one of its orbits on M is open. It is easy to see that the study of decompositions of real Lie algebras is equivalent to the study of inclusion relations among locally transitive Lie groups of transformations. If G is a complex Lie algebra and G' and G'' are its complex subalgebras, then the decomposition (G, G', G'') is called **complex**. A decomposition is called **semisimple** if G, G', and G'' are semisimple. The present paper is devoted to finding all real and complex semisimple decompositions.

An essential role here will be played by the following circumstance: if a semisimple subgroup $\mathfrak{G}'$ of a semisimple Lie group $\mathfrak{G}$ is locally transitive on the manifold $\mathfrak{G}/\mathfrak{G}''$, where $\mathfrak{G}''$ is also semisimple, then $\mathfrak{G}'$ is transitive on $\mathfrak{G}/\mathfrak{G}''$. This result makes it possible to reduce the study of real and complex semisimple decompositions to the study of decompositions of compact Lie algebras. Decompositions of the latter were completely studied in (^{1}) (for a detailed exposition see (^{2})).

1. A complex Lie group is called **reductive** if it is the complex envelope of some connected compact Lie group (see (^{3})).

Theorem 1. *Let $\mathfrak{G}$ be a reductive complex Lie group, and let $\mathfrak{G}'$ and $\mathfrak{G}''$ be its reductive subgroups. If $\mathfrak{G}'$ is locally transitive on the manifold $Z = \mathfrak{G}/\mathfrak{G}''$, then it is transitive on Z.*

Proof. If $\mathfrak{G}'$ is locally transitive on Z, then its open orbit is dense in Z. Consequently, every continuous and, in particular, holomorphic function on Z that is invariant with respect to $\mathfrak{G}'$ is constant. Suppose that $\mathfrak{G}'$ is not transitive on Z, and let us construct on Z a nonconstant holomorphic function invariant with respect to $\mathfrak{G}'$. For this purpose we first show that $\mathfrak{G}'$ has at least two distinct closed orbits on Z. Let $\mathfrak{K}$ be a maximal compact subgroup of the group $\mathfrak{G}$ containing a maximal compact subgroup $\mathfrak{K}''$ of the group $\mathfrak{G}''$, and let $\mathfrak{K}'$ be an arbitrary maximal compact subgroup in $\mathfrak{G}'$. Then there exist elements $g \in \mathfrak{G}$ such that $g^{-1}\mathfrak{K}'g \subset \mathfrak{K}$. From the results of (^{3}) it follows that, if g is such an element, then the orbit $\mathfrak{G}'(go)$ is closed in Z. If we fix one element g_0 possessing the property indicated above, then all elements of the form g_0k, where $k \in \mathfrak{K}$, will possess the same property. Consequently, the orbits of all points of the manifold $g_0\mathfrak{K}(o)$ with respect to $\mathfrak{G}'$ will be closed. If $\mathfrak{G}'$ has on Z only one closed orbit S, then $g_0\mathfrak{K}(o) \subset S$, whence it follows easily that $\dim S = \dim Z$ and $S = Z$, i.e., $\mathfrak{G}'$ is transitive. Thus, $\mathfrak{G}'$ has two distinct closed orbits S_0 and S_1. Since Z is a Stein manifold (^{3}), there exists on Z a holomorphic function f equal to 0 on S_0 and to 1 on S_1. Consider the function

$$\tilde{f}(z) = \int_{\mathfrak{K}'} f(kz)\, dk,$$

Table 1

G	G'	i'	G''	i''	U
A_{2n-1}^{2l}	C_n^l	φ_1	$\dfrac{A_{2n-2}^{2l}}{A_{2n-2}^{2l-1}}$	$\varphi_1 + N$	$\dfrac{C_{n-1}^l}{C_{n-1}^{l-1}}$
I_{2n-1}	IC_n	φ_1	I_{2n-2}	$\varphi_1 + N$	IC_{n-1}
B_3^3	G_2^1	φ_2	$\dfrac{D_3^2}{D_3^3 = I_3}$	$\varphi_1 + N$	$\dfrac{A_1^2}{I_2}$
B_3^3	G_2^1	φ_2	$\dfrac{B_2^1}{B_2^2 = IC_2}$	$\varphi_1 + 2N$	$\dfrac{A_1^0}{A_1^1 = IC_1}$
D_{n+1}^{2l}	$\dfrac{B_n^{2l}}{B_n^{2l-1}}$	$\varphi_1 + N$	A_n^l	$\varphi_1 + \varphi_n$	$\dfrac{A_{n-1}^l}{A_{n-1}^{l-1}}$
D_{n+1}^{n+1}	B_n^n	$\varphi_1 + N$	I_n	$\varphi_1 + \varphi_n$	I_{n-1}
D_{2n}^{4l}	B_{2n-1}^{4l}	$\varphi_1 + N$	$\dfrac{C_n^l}{C_n^l \oplus C_1^0}$	$\varphi_1 + \varphi_1$	$\dfrac{C_{n-1}^l}{C_{n-1}^l \oplus C_1^0}$
D_{2n}^{4l}	B_{2n-1}^{4l-1}	$\varphi_1 + N$	$\dfrac{C_n^l}{C_n^l \oplus C_1^0}$	$\varphi_1 + \varphi_1$	$\dfrac{C_{n-1}^{l-1}}{C_{n-1}^{l-1} \oplus C_1^0}$
D_{2n}^{2n}	B_{2n-1}^{2n-1}	$\varphi_1 + N$	$\dfrac{IC_n}{IC_n \oplus IC_1}$	$\varphi_1 + \varphi_1$	$\dfrac{IC_{n-l}}{IC_{n-1} \oplus IC_1}$
D_8^8	B_7^7	$\varphi_1 + N$	$\dfrac{B_4^1}{B_4^4}$	φ_4	$\dfrac{B_3^0}{B_3^3}$
D_4^4	B_3^3	φ_3	B_3^3	$\varphi_1 + N$	G_2^1
D_4^4	B_3^3	φ_3	$\dfrac{D_3^2}{D_3^3 = I_3}$	$\varphi_1 + 2N$	$\dfrac{A_2^1}{I_2}$
D_4^4	B_3^3	φ_3	$\dfrac{B_2^1}{B_2^1 \oplus A_1^0}$	$\varphi_1 + 3N$	$\dfrac{A_1^0}{A_1^0 \oplus A_1^0}$
D_4^4	B_3^3	φ_3	$\dfrac{B_2^2 = IC_2}{B_2^2 \oplus A_1^1}$	$\varphi_1 + 3N$	$\dfrac{A_1^1 = IC_1}{A_1^1 \oplus A_1^1}$

where integration is carried out with respect to the invariant measure on $\mathfrak{K}'$, normalized by the condition

$$\int_{\mathfrak{K}'} dk = 1.$$

Obviously, $\tilde{f}$ is holomorphic and invariant with respect to $\mathfrak{G}'$. On the other hand, it is clear that $\tilde{f} = 0$ on S_0 and $\tilde{f} = 1$ on S_1, so that f is not constant.

A complex Lie algebra G is called **reductive** if $G = K^C$, where K is some compact Lie algebra. A decomposition (G, G', G'') is called **reductive** if G, G', and G'' are reductive.

Corollary of Theorem 1. *Let* (G, G', G'') *be a reductive complex decomposition. Then the corresponding triple of reductive complex Lie groups is also a decomposition. Furthermore, there exists a decomposition* (K, K', K'') *of the compact Lie algebra* K *and an inner automorphism* A *of the algebra* G *such that* $G = K^C$, $G' = A({K'}^C)$, $G'' = K''^C$.

This corollary shows that reductive complex decompositions are in one-to-one correspondence with decompositions of compact Lie algebras. Therefore the description of all decompositions of compact Lie algebras given in Theorems 1, 2, 3 of the paper (1) is applicable also to reductive (in particular, to semisimple) complex decompositions.

Relying on Theorem 1, one can prove an analogous theorem for the real case.

Theorem 2. *Let* $\mathfrak{G}$ *be a connected semisimple Lie group, and let* $\mathfrak{G}'$ *and* $\mathfrak{G}''$ *be its connected semisimple subgroups. If* $\mathfrak{G}'$ *is locally transitive on the manifold* $\mathfrak{G}/\mathfrak{G}''$*, then it is transitive on* $\mathfrak{G}/\mathfrak{G}''$*.*

2. The results obtained make it possible to give a complete description of real semisimple decompositions. Obviously, with each such decomposition (G, G', G'') there is associated a complex semisimple decomposition $(G^C, {G'}^C, G''^C)$. Using Theorem 2, one can show that the converse assertion is also true: if (G, G', G'') is such a triple of semisimple algebras that $(G^C, {G'}^C, G''^C)$ is a decomposition, then $G = G' + G''$.

Let us first assume that G is simple. Then two cases are possible:
1) G^C is simple; 2) G^C is not simple and G is a simple complex Lie algebra considered as real. Using the results of the paper (1), as well as the results of F. I. Karpelevich (4) on subalgebras of real semisimple Lie algebras, we arrive at the following theorem.

Theorem 3. *All semisimple decompositions* (G, G', G'') *of noncompact simple Lie algebras* G *of the first type are listed in Table 1 (the subalgebras* G' *and* G'' *are indicated up to conjugacy, and from each pair of triples* (G, G', G'') *and* (G, G'', G') *only one is given).*

In Table 1, i' and i'' denote the embeddings $G' \to G$ and $G'' \to G$, and in the corresponding columns the linear representations of the algebras ${G'}^C$ and G''^C realizing these embeddings are indicated. The notation of papers (1,2) is used here. In the last column the subalgebras $U = G' \cap G''$ are given. The real forms are denoted as follows: A_n^l (respectively C_n^l) is the real form of the algebra A_n (respectively C_n) consisting of matrices that leave invariant a nondegenerate

Hermitian form with negative index of inertia l (respectively $2l$); B_n^l and D_n^l are the algebras of real matrices of orders $2n+1$ and $2n$, leaving invariant a nondegenerate quadratic form with negative index of inertia l; I_n and IC_n are the real forms of the algebras A_n and C_n, consisting of real matrices; G_2^0 and G_2^1 are, respectively, the compact and noncompact real forms of the algebra G_2.

In the second case the following is true.

Theorem 4. *Let G be a simple complex Lie algebra, considered as an algebra over the field of real numbers, and let (G, G', G'') be its decomposition, with G' and G'' simple. Then, with the exception of two cases, G' and G'' are complex subalgebras in G. The exceptional decompositions have the following form: $G = D_4$, $G' = B_3$ (the spinor subalgebra), $G'' = D_4^1$ or D_4^3, $U = G_2^0$ or G_2^1, respectively.*

We now pass to the description of arbitrary semisimple decompositions. A Lie algebra is called **strongly semisimple** if it is semisimple and contains no ideals of rank 1. A decomposition (G, G', G'') is called **strongly semisimple** if G, G', and G'' are strongly semisimple. From Theorem 2 of paper $(^1)$ it follows that it is enough to restrict ourselves to the study of strongly semisimple decompositions.

Theorem 5. *All irreducible effective strongly semisimple decompositions can be obtained from decompositions of simple Lie algebras by means of the construction set forth on p. 263 of paper* $(^1)$.

3. Let us consider one application of the results obtained. Let $M = \mathfrak{G}/\mathfrak{U}$, where $\mathfrak{G}$ is a connected real Lie group, $\mathfrak{U}$ is its connected closed subgroup, and let G and U be the Lie algebras of the groups $\mathfrak{G}$ and $\mathfrak{U}$. In order that there exist on M a complex structure invariant with respect to $\mathfrak{G}$, it is necessary and sufficient that there exist a complex subalgebra $H \subset G^C$ such that $G^C = G + H$, and $G \cap H = U$ $(^5)$. Thus, the determination of all complex homogeneous spaces of the group $\mathfrak{G}$ reduces to the determination of certain special decompositions of the algebra G^C.

We shall consider the case when G is semisimple, and assume additionally that H is also semisimple. It can be shown that the latter condition

* By D_4 and B_3 here and in what follows are denoted the complex Lie algebras of types D_4 and B_3.

is always satisfied if M admits an invariant complex structure and U is not contained in any non-semisimple subalgebra of the algebra G. From Theorem 2 it follows that in the case under consideration M is analytically homeomorphic to the manifold $\mathfrak{G}^C/\mathfrak{H}$, where $\mathfrak{G}^C$ is the complex Lie group associated with the algebra G^C, and $\mathfrak{H}$ is its semisimple complex subgroup.

Theorem 6. *In the case under consideration, M admits a complex structure invariant with respect to $\mathfrak{G}$ if and only if the Lie algebras G and U have the following form:*

$$G = D_4^R \oplus P, \qquad U = G_2^R \oplus Q.$$

Here $D_4^R = D_4^1$ *or* D_4^3, $G_2^R = G_2^0$ *or* G_2^1, *respectively;* P *is a semisimple complex Lie algebra containing a complex subalgebra* P' *isomorphic to* B_3; Q *is its complex subalgebra lying in the centralizer of the subalgebra* P'; G_2^R *is naturally monomorphically projected onto both summands of the algebra* G*, and the projection onto* P *is contained in* P'.

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