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Abstract

Full Text

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ON ESSENTIAL AND ZERO-DIMENSIONAL MAPPINGS*

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For compacta (more general cases are indicated below) the following two theorems from dimension theory are known**.

A. Theorem of P. S. Aleksandrov. The inequality $\dim X \geq k$ holds if and only if the compactum X possesses an essential*** mapping into the k-dimensional cube.

G. Theorem of Hurewicz. The reverse inequality $\dim X \leq k$ holds if and only if the compactum X possesses a zero-dimensional**** mapping into the k-dimensional cube.

From this we immediately obtain that the existence of a pair of mappings of the compactum X into one and the same k-dimensional cube, of which the first mapping is essential and the second zero-dimensional, is a necessary and sufficient condition for the compactum X to have dimension k.

Remaining within the framework of the classical definitions, one may say that the aim of the present note is to obtain a stronger necessary (and at the same time sufficient) condition, namely:

Theorem 1. *Every compactum of dimension k possesses an essential (and at the same time) zero-dimensional mapping into the k-dimensional cube.*

Corollary. *A compactum has dimension k if and only if it possesses an essential zero-dimensional mapping into the k-dimensional cube.*

Theorem 1 follows easily from the following fact:

Theorem 2. *For every normal space of dimension k, each of its closed zero-dimensional mappings into the k-dimensional cube is locally essential*****.*

Definition 1. We shall call a mapping f of a space X into a space Y **strongly refinable** if it is γ-zero-dimensional****** for every finite open covering γ of the space X.

Under suitable separation conditions, *every closed zero-dimensional mapping will be strongly refinable, and every strongly refinable mapping will be refinable******. Under strongly refinable mappings, dimension does not decrease.*

* We consider only nonempty topological spaces and only continuous mappings.

** By the dimension of a topological space we mean here only the dimension defined by means of finite open coverings.

*** A mapping f of a space X into a cube I is called **essential** if one cannot find a mapping of the space X into the boundary S of the cube I that would coincide with the mapping f on the full preimage $f^{-1}S$ of the boundary S.

**** A mapping f of a space X into a space Y is called **zero-dimensional** if $\sup\{\dim f^{-1}y\} = 0$, $y \in Y$.

***** A mapping f of a space X into a finite-dimensional polyhedron P (not necessarily finite) will be called **locally essential** if it is essential on the full preimage of some simplex (or cube) of highest dimension lying in P.

****** A mapping f of a space X into a space Y is called **γ-zero-dimensional**, where γ is a covering of the space X, if there exists such an open locally finite covering ω of the space Y that the full preimage of every element of the covering ω splits into a sum of open pairwise disjoint sets, with the system of which inscribed in γ.

******* A mapping f of a space X into a space Y is called **refinable** if for every point x of the space X and for every neighborhood Ox of it there exists in the space Y an open set whose full preimage splits into the sum of two open disjoint sets G and H such that $x \in H \subset Ox$.

Theorem 3. *A strongly light mapping of a normal space into a finite-dimensional polyhedron is locally essential if and only if it preserves dimension.*

Indeed, let γ be an arbitrary finite open covering of the space X. By virtue of the paracompactness of the polyhedron P and the γ-zero-dimensionality of the given strongly light mapping f, there is a simplicial subdivision K of the polyhedron P such that the full inverse image $f^{-1}OOa$ of the "double" star OOa** of each vertex a of the subdivision K decomposes into a sum of pairwise disjoint open sets whose collection is inscribed in the covering γ.

If the mapping f is locally inessential, then it can be changed on the full inverse images of all simplexes of higher dimension of the subdivision K (even by means of a deformation), without changing it on the full inverse images of all simplexes of lower dimension, so that for the changed mapping g the point gx belongs to the boundary of a simplex of higher dimension, if fx belonged to this simplex itself. Then, taking an open covering $\omega = \{U\}$ of the sum of all simplexes of dimension $< k$, where $k = \dim P$, so that it is inscribed in the covering $\omega' = \{Oa\}$ (of open stars) and has multiplicity $\leq k$, we obtain the open covering $\{g^{-1}U\}$ of the space X, inscribed in γ and having multiplicity $\leq k$. Hence $\dim X < k = \dim P$, if f is locally inessential. The converse assertion is known.

In a similar way one obtains

Theorem 3′. *For every normal space X of dimension k, the set of all its* *locally essential mappings into the k-dimensional cube I is open and dense in the mapping space $C(X, I)$.*

Hence, as consequences, we obtain two theorems:

Theorems 4 and $4'$. *For every normal space X admitting a light mapping into Hilbert cube**** *(respectively, for every metric space), the set of all its locally essential light (respectively, strongly zero-dimensional******) mappings into the cube I of the same dimension is dense in the space $C(X, I)$.*

From this there follow immediately the natural corollaries on the existence of locally essential light and locally essential strongly zero-dimensional mappings under the conditions indicated here. Of course, for the proof of Theorems 4 and $4'$ it is first necessary to prove that the set of all light mappings and the set of all strongly zero-dimensional mappings are everywhere dense in the space $C(X, I)$. The first was in fact done in (1^a) (see Theorem 2 on the existence of light mappings), and the second—earlier by M. Katětov in (2).

Remark 1. Since for paracompact locally bicompact spaces the notions of a light and a zero-dimensional mapping are equivalent (see (1^a)), and since every quotient space of any locally bicompact group (in particular, every such group) is paracompact, in the assertion of Theorem 4, in the case of paracompact locally bicompact Z-spaces (in particular, for finite-dimensional quotient spaces of locally bicompact groups), one may consider zero-dimensional mappings instead of light mappings.

* By the **double star** OOa of a vertex a of the subdivision K we shall mean the sum of the open stars of all vertices of the closure of the open star Oa.

** These spaces were first considered by A. Zarelua in (1^a) and called Z-spaces. It is proved there that every finite-dimensional quotient space of every locally bicompact group is a Z-space. As follows from the next footnote, every metric space is also a Z-space.

*** A mapping f of a metric space X into a metric space Y is called **strongly zero-dimensional** if for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that, whenever the diameter of an open set H of the space Y is less than δ, the full inverse image $f^{-1}H$ decomposes into a sum of pairwise disjoint open sets, each of which has diameter less than ε. It is easy to see that every strongly zero-dimensional mapping is light.

Remark 2. Strongly light and even closed zero-dimensional mappings are not as good as light mappings or strongly zero-dimensional mappings. Even one-dimensional spaces with a countable base may have no strongly light mapping into an interval at all; such a space turns out to be the subset of all those points of the square whose second coordinate is rational (the product of an interval by the set of rational points). This example for closed zero-dimensional mappings was considered by A. Taĭmanov in ([3]).

Remark 3. The assertion of Theorem 3 remains true if, instead of the dimension dim, one considers the inductive dimension of Uryson–Menger, and instead of strongly light mappings–light mappings, but only for the dimensions $\operatorname{ind} X = 0, 1$. In the general case this is not clear even for strongly zero-dimensional mappings of spaces with a countable base.

In order to obtain a dimension characteristic from Theorems 4 and 4′ for non-compact spaces, one must have, for the corresponding cases, a theorem of the type of Hurewicz' s theorem. For metric spaces and strongly zero-dimensional mappings such a theorem was proved by M. Katětov in ([2])*. For light mappings and the Uryson–Menger dimension, a similar assertion was proved by A. Zarelua in ([1]). He also proved that for strongly paracompact spaces admitting light mappings into Hilbert' s cube, the dimensions dim and ind coincide (see ([1])). He considers even a broader class of completely paracompact spaces (see ([1] ,)). Therefore we have:

Corollary.** *The dimension of a completely paracompact space admitting a light mapping into Hilbert' s cube (respectively, the dimension of a metric space) is equal to k if and only if it possesses a light (respectively, strongly zero-dimensional) essential mapping into the k-dimensional cube.*

Theorem 5. *For every light mapping f of a space X with a countable base into a completely bounded metric space Y, there exists a metric of the space X in which the mapping f is strongly zero-dimensional.*

Indeed, by a theorem of A. Zarelua, for the (compact) completion cY of the space Y one can choose such a compact extension aX (with a countable base) of the space X on which the given mapping f is continued to a zero-dimensional mapping $\tilde{f}$ (see ([1])).

By virtue of a theorem of M. Shershnev ([4]), this mapping $\tilde{f}$ is strongly zero-dimensional with respect to any metric of the compactum aX. Hence, for some metric of the space X, the mapping f is strongly zero-dimensional.

Corollary. *A mapping f of a space X with a countable base into a space Y with a countable base is light if and only if, with respect to some metrics of these spaces, it is strongly zero-dimensional.*

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[3] A. Taĭmanov, UMN, **15**, No. 5, 187 (1960).
[4] M. Shershnev, Matem. sborn., **60**, No. 2 (1963).

* In the same work M. Katětov was the first to consider strongly zero-dimensional mappings. He calls them uniformly zero-dimensional.

** *Note added in proof.* In Bull. Am. Math. Soc. an example appeared of a metric space M such that $\operatorname{ind} M = 0$, but $\dim M = 1$. Therefore the first assertion of our corollary (as well as Theorem 5 and its corollary) is false for any metric spaces.

Note: Figure translations are in progress. See original paper for figures.