



Soviet-era science, translated into English

MATHEMATICS

V. PONOMAREV

1963

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196301.74279>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

V. PONOMAREV

ON THE ABSOLUTE OF A TOPOLOGICAL SPACE

(Presented by Academician P. S. Aleksandrov on 20 XI 1962)

1. Let us call a **preimage** of a topological space X any space X' such that there exists an irreducible perfect mapping* f of the space X' onto X ; in ⁽²⁾, for every paracompact X , I constructed a space, which I called the absolute of the space X , and which is characterized by the property that it is the unique maximal element in the partially ordered set of all preimages of the space X .

The absolute $\mathfrak{w}X$ of a paracompact X is always a completely zero-dimensional space, in the sense that into every open covering of the space $\mathfrak{w}X$ one can inscribe a covering consisting of disjoint open (hence, open-and-closed) sets.

Since every completely zero-dimensional space is paracompact, it follows from the existence for a space X of a completely zero-dimensional absolute (and, in general, of a completely zero-dimensional preimage) that X is paracompact; thus, in ⁽²⁾ a new characterization of paracompacts was obtained.

2. The direct construction of the absolute of a space X given by me in ⁽²⁾ is based on the projection spectrum of the space X corresponding to the system of all locally finite decompositions of this space. After my work, various authors (S. Iliadis, J. Flachsmeyer, and others) proposed other methods of constructing the absolute and extended this concept to broader classes of topological spaces (naturally, with the abandonment of complete zero-dimensionality of the absolute). A particularly successful construction of the absolute was proposed by S. Iliadis ⁽⁶⁾, who applied for this purpose the method of centered systems of open sets (going back to P. S. Aleksandrov ⁽³⁶⁾ and later considered by S. V. Fomin ⁽⁴⁾ and by me ⁽⁵⁾). By this method S. Iliadis constructed the absolute for any regular and even for any Hausdorff space; however, in the latter case, in the definition of a perfect mapping it is necessary to replace continuity by θ -continuity in the sense of S. V. Fomin.
3. In the present paper I first of all show that the fully carried-out construction of the absolute for any Hausdorff space is already contained in my work ⁽¹⁾. Indeed, the system $\mathfrak{A} = \{\alpha\}$ of all finite decompositions**

$\alpha = \{A_1^\alpha, \dots, A_{s_\alpha}^\alpha\}$ of any regular space X is directed (in the natural way: $\alpha' > \alpha$, if α' is inscribed in α)

* All spaces considered in this note are Hausdorff. A single-valued mapping f of a space X onto a space Y is called **irreducible** if for every closed set $A \subset X$, $A \neq X$, we have $fA \neq Y$. A mapping f is called **perfect** if it is: 1) continuous; 2) closed; 3) bicomact in the sense that the preimages of all $y \in Y$ (and, in the case of multivalued mappings, also the preimages of all $x \in X$) are bicomacts. A mapping f is called **θ -continuous at a point $x \in X$** ⁽⁴⁾, if for every neighborhood Oy of the point $y = fx$ there exists a neighborhood Ox such that $fOx \subset [Oy]$. A multivalued mapping f is called **irreducible** if there exist a space Z and irreducible single-valued mappings $f_X : Z \rightarrow X$ and $f_Y : Z \rightarrow Y$ such that $f = f_Y f_X^{-1}$; moreover, if f is perfect, then it is required that f_X and f_Y also be perfect.

** In ⁽¹⁾, a **decomposition** of a space X is called a finite covering α , whose elements are the closures of pairwise disjoint open sets. In ⁽²⁾ I replace the requirement that the covering α be finite by the broader requirement of local finiteness. In the present paper only finite decompositions are again considered,

and is a refinement (for every point $x \in X$ and its neighborhood Ox there exists an α such that the star of the point x with respect to the cover α is contained in Ox); if X is any Hausdorff (possibly nonregular) space, then one can only assert that the system \mathfrak{A} is θ -refining in the sense that for every $x \in X$, Ox , there exists an α such that the star of the point in the cover α is contained in the closure $[Ox]$ of the neighborhood Ox . The cardinality τ of the set \mathfrak{A} does not exceed 2^m , where m is the weight of the space X .

In the paper ⁽¹⁾, for each $\alpha = \{A_1^\alpha, \dots, A_{s_\alpha}^\alpha\}$ one considers the finite set

$$D_\alpha = \{1_\alpha, 2_\alpha, \dots, s_\alpha\}$$

of natural numbers provided with the index α ; for $\alpha' > \alpha$ a projection $\mathfrak{P}_\alpha^{\alpha'} : D_{\alpha'} \rightarrow D_\alpha$ is defined (we put $\mathfrak{P}_\alpha^{\alpha'} i_{\alpha'} = i_\alpha$, where i_α is the number of the unique element $A_{i_\alpha}^\alpha$ containing the given element $A_j^{\alpha'} \in \alpha'$). Thus we obtain an inverse spectrum $\{D_\alpha, \mathfrak{P}_\alpha^{\alpha'}\}$ with limit space \bar{D} , which is a bicomactum lying in the topological product $D^\tau = \prod_{\alpha \in \mathfrak{A}} D_\alpha$; the cardinality of D^τ is $2^\tau \leq 2^{2^m}$. We shall call a point $\xi = \{i_\alpha\} \in \bar{D} \subseteq D^\tau$ **marked** if $\bigcap_\alpha A_{i_\alpha}^\alpha \subset X$ is nonempty (and then, even in any Hausdorff X , it consists of one point $x = \pi_X(\xi)$). Denote by $\omega X \subseteq \bar{D}$ the set of all marked points of the bicomactum \bar{D} ; the set ωX (considered as a subspace of the bicomactum \bar{D}) is the **absolute of the space X** .

Indeed, the following holds:

Main theorem 1. *Let X be a Hausdorff space of weight m . Then the completely regular space ωX , uniquely determined by the space X , of cardinality $\leq 2^{2^m}$ and weight $\leq 2^m$, is extremally disconnected (and, consequently, zero-dimensional in the sense $\text{ind } \omega X = 0$); it is mapped onto X by means of the (defined above) irreducible, bicomact, closed θ -continuous mapping $\pi_X : \omega X \rightarrow X$; moreover, every irreducible closed continuous mapping h of some completely regular space Z onto ωX is a homeomorphism.*

If X is regular, then the mapping $\pi_X : \omega X \rightarrow X$ is continuous (i.e. perfect) and ωX is a preimage of the space X ; moreover, ωX is the unique maximal perfect preimage of the space X (i.e. the space ωX is uniquely determined by the condition that it is a preimage of every preimage of the space X).

Two (regular) spaces X and Y have homeomorphic absolutes if and only if one of them can be irreducibly and perfectly (but, generally speaking, multivalently) mapped onto the other; every such mapping (in particular, every single-valued mapping $f : X$ onto Y) is given by the formula

$$f = \pi_Y \hat{f} \pi_X^{-1}, \quad (1)$$

where \hat{f} is some homeomorphism of the absolute ωX onto the absolute ωY ; conversely, every homeomorphism $\hat{f} : \omega X$ onto ωY determines by formula (1) a perfect (multivalued) mapping $f : X$ onto Y .

In the case of a paracompact X , the definition of the absolute ωX given in the present note on the basis of finite partitions is equivalent to the definition (given in note (2)) on the basis of locally finite partitions.

*Finally, a Hausdorff space is an absolute (of some Hausdorff space) if and only if it is completely regular and extremally disconnected**; in this case it is the absolute of itself.*

* In the paper (1) the continuation $\bar{f} : \bar{D} \rightarrow \bar{X}$ of the mapping π is defined, the existence of which is asserted by Theorem 3 of the preceding note by S. Iliadis.

** That is, every canonical closed set in it is open.

The estimates of the cardinality and weight of the absolute cannot be improved. B. Efimov showed that the absolute of every countable compactum is the Čech extension βN of a space N consisting of a countable set of isolated points.

4. We proceed to the proof. First of all, it is proved that the mapping defined above

$$\pi_X : \omega X \rightarrow X$$

(which consists in the fact that to each point $\xi = \{i_\alpha\} \in \omega X \subseteq \bar{D}$ we assign the point

$$\pi_X \xi = \bigcap_{\alpha} A_{i_\alpha}^\alpha \in X$$

) is a bicomact, closed, irreducible mapping, θ -continuous in the case of an arbitrary Hausdorff X , and continuous in the case of a regular X . The proof differs from the detailed proof carried out in (1) only in that now (following the notation of (1)) by \bar{X} one must understand some bicomact T_1 -extension of the space X (for example, the Wallman extension).

The fact that in the case of an arbitrary Hausdorff, respectively regular, X , the mapping π_X is θ -continuous, respectively continuous, follows from the property of the system \mathfrak{A} to be θ -refining, respectively refining. In the rest the proof literally repeats the proof given in (1).

The estimates of the cardinality and weight of the absolute follow directly from its construction.

We now prove (by the methods of my papers (1, 2)) the following auxiliary propositions.

Proposition 1. *The space ωX is extremally disconnected.*

Proof. First of all let us note that, by the closedness and irreducibility of the mapping $\pi_X = \pi$, for every open $H \subseteq \omega X$ the set

$$\pi^\# = \mathcal{E}(x \in X, \pi^{-1}x \subseteq H)$$

is open and nonempty. Moreover, $\pi[H] \supseteq [\pi^\#H]$. Consider in the space X the partition $\alpha = \{A_1, A_2\}$, where

$$A_1 = [\pi^\#H], \quad A_2 = [X \setminus [\pi^\#H]].$$

In the notation of (1) we have, obviously,

$$U_1^\alpha \subseteq \pi^{-1}A_1 \subseteq \pi^{-1}[\pi^\#H] \subseteq \pi^{-1}\pi[H]. \quad (2)$$

We shall show that $[H] = U_1^\alpha$, and this will prove assertion 1. But

$$H \subseteq J\pi^{-1}A_1 \subseteq U_1^\alpha,$$

and since U_1^α is open-and-closed, it follows that $[H] \subseteq U_1^\alpha$. Suppose that

$$U_1^\alpha \setminus [H] \neq \Lambda.$$

Since π is irreducible and $U_1^\alpha \setminus [H]$ is open, there exists a point $x_0 \in X$ such that

$$\pi^{-1}x_0 \subseteq U_1^\alpha \setminus [H],$$

which contradicts formula (2), according to which

$$\pi^{-1}x_0 \cap [H] \neq \Lambda.$$

The assertion is proved.

Remark. In the same way one proves the extremality of the bicomactum \bar{D} (containing ωX as an everywhere dense set).

We now prove

Proposition 2. *Every extremally disconnected completely regular space X is its (own) absolute: $\omega X = X$.*

Indeed, let us prove that every perfect mapping

$$\pi : \omega X \rightarrow X$$

is a homeomorphism.

We first prove that π is open. Since ωX is (completely) regular and extremal, the canonical closed sets $[H]$ in ωX are open and form a base of the space ωX . Therefore it is enough to prove that $\pi[H]$ is open in X (for every open $H \in \omega X$). By the closedness and irreducibility of the mapping π we have

$$\pi[\overline{H}] = [\pi^\# H].$$

But $\pi^\# H$ is open in X , hence (since X is extremal) it is open also that

$$[\pi^\#] = \pi[H].$$

Thus, the mapping π is open; but an open mapping π is irreducible; consequently it is one-to-one, and, being closed, is topological. Assertion 2 is proved.

The remaining assertions of Theorem 1 follow from the main lemma of my paper (2), whose proof is preserved without change in the case

of arbitrary regular spaces; a consequence of the main lemma of paper (2) is the validity of formula (1) and of Theorems 4 and 5 of paper (2) for the case of regular X and Y , which completes the proof of our main theorem.

5. For a completely regular space X the following formula holds (first proved by S. Iliadis)

$$\omega\beta X = \beta\omega X.$$

Indeed, take $\overline{X} = \beta X$; then the mapping $\overline{\pi} : \overline{D} \rightarrow \overline{X}$ (the mapping \overline{f} of paper (1)) is irreducible, and hence $\omega\beta X = \omega\overline{D} = \overline{D}$ (the latter because \overline{D} is extremally disconnected). On the other hand, there exists a natural irreducible mapping $h : \beta D \rightarrow \overline{D}$, which (since $\overline{D} = \omega D$) is a homeomorphism; thus,

$$\overline{D} = \beta D = \beta\omega X,$$

and formula (3) is proved.

Remark 1. Every dense subset X of an extremally disconnected space X' is itself extremally disconnected.

Proof. Let F be a canonical closed set in X ; then $[F]_{X'} = \Phi$ is a canonical closed set in X' and, consequently (since X' is extremally disconnected), is

open-and-closed in X' ; but then $F = X \cap \Phi$ is an open-and-closed set in X , as was required to prove.

Remark 2. In the left-hand side of formula (1), the space βX may be replaced by any Hausdorff bicomact extension bX of the space X (all of them have the same absolute). In particular, for the space N , consisting of a countable set of isolated points, we obtain (noting that $\omega N = N$) B. Efimov's theorem

$$\omega bN = \beta N$$

for any bN .

6. For normal spaces X the following spectral construction of the space $\omega\beta X = \beta\omega X$ is obtained: if X is normal, then the nerves $|\alpha|$ of all finite partitions α with the natural projections $\delta_\alpha^{\alpha'}$ form a spectrum

$$S = \{|\alpha|, \delta_\alpha^{\alpha'}\},$$

called the spectrum of the normal space X . The limit space of the spectrum S is the Čech bicomact extension βX , and S is the (maximal) spectrum of the space βX . On the other hand, the spectrum $\{D_\alpha, \delta_\alpha^{\alpha'}\}$ considered above is nothing other than the full refinement ωS of the spectrum S in the sense of ⁽²⁾. Its limit space, i.e. our bicomact \bar{D} , is therefore the absolute of the space βX .

Theorem 2*. *The space of the full refinement ωS of the spectrum S of a normal space X is the absolute of the space βX . But the zero-dimensional spectrum ωS , being based on all finite partitions of the space ωX , has as its limit the space $\beta\omega X$ —we have again obtained formula (3).*

Moscow State University
named after M. V. Lomonosov

Received
6 XI 1962

CITED LITERATURE

- ¹ V. Ponomarev, DAN, **132**, No. 6, 1269 (1960).
- ² V. Ponomarev, DAN, **143**, No. 1, 46 (1962).
- ³ P. Aleksandrov: a) UMN, **2**, 1 (17), 5 (1947); b) Matem. sborn., **5** (47), 403 (1939).
- ⁴ S. Fomin, Ann. of Math., **44**, 471 (1943).
- ⁵ P. Aleksandrov, V. Ponomarev, DAN, **121**, No. 4, 575 (1958).
- ⁶ S. Iliadis, DAN, **149**, No. 1 (1963).

* It should be noted that for a Hausdorff H -closed space X we have $\omega X = \bar{D}$, i.e. the absolute ωX is a bicomactum.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.